JOURNAL OF ALGEBRA 138, 440–478 (1991)

The Projective Characters of the Symmetric Groups that Remain Irreducible on Subgroups*

PETER B. KLEIDMAN AND DAVID B. WALES

Department of Mathematics, California Institute of Technology, Pasadena, California 91125

Communicated by Gordon James

Received June 7, 1989

1. INTRODUCTION

An important problem in studying embeddings of one finite group in another is to determine all subgroups H of a group G for which an irreducible character of G remains irreducible upon restriction to H. Such questions arise, for example, in the study of maximal subgroups of the finite classical groups, where one seeks to classify triples $H < G < GL(V)$, where both H and G are quasisimple and act absolutely irreducibly in $GL(V)$ (see Problem 4 in [1] and Problems 4 and 5 in [21], for example).

This problem has been solved when G is the symmetric group S_n or the alternating group A_n by Saxl [17]. In this paper we attack the same problem for faithful characters of a non-splitting double cover $\hat{S}_n$ of S_n. For each $n \geqslant 4$, there are two such groups [18]. Their irreducible representations, however, are very similar. Indeed, given a set of matrices in an irreducible representation of one double cover, we obtain a representation of the other by multiplying certain matrices by the scalars $\pm i$, where $i = \sqrt{-1}$. Consequently, subgroups for which a faithful character remains irreducible correspond in the two double covers. So with no loss of generality, we choose a particular $\hat{S}_n$, following [22], which applies for all n. Here each transposition in S_n lifts to an element of order 4 in $\hat{S}_n$, and disjoint transpositions in S_n generate a quaternion group Q_8 in $\hat{S}_n$.

It turns out that finding all subgroups of $\hat{S}_n$ upon which a faithful character remains irreducible is a rather unwieldy task. Consequently, we restrict our attention to the maximal subgroups of $\hat{S}_n$. This result appears in Theorem 1.1. In Theorem 1.2 we obtain the analogous result for $\hat{A}_n$. Theorem 1.3 classifies those quasisimple subgroups of $\hat{A}_n$ for which an irreducible character remains irreducible—this has direct application to the

* The authors were partially supported by an NSF grant.

0021-8693/91 $3.00

problem mentioned in the first paragraph. Finally, Theorem 1.4 provides the corresponding results for the three-fold and six-fold covers of A_6 and A_7. While Theorems 1.1 and 1.2 are concerned only with maximal subgroups, several results in this paper can be used to obtain information about the restrictions of characters to non-maximal subgroups. For example, Theorems 6.3 and 6.4 classify all instances in which a character of $\hat{S}_n$ or $\hat{A}_n$ remains irreducible upon restriction to a primitive subgroup. Further, Theorem 5.1 imposes strong conditions on a character of $\hat{S}_n$ which can remain irreducible upon restriction to an imprimitive subgroup.

The group $\hat{S}_n$ was first investigated in detail by Schur [18], who determined the characters and found the character tables using certain functions defined inductively. To describe these characters, we first need to define the set P_n of *partitions of n*. A partition λ of n is a sequence $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_l\}$, such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$ and $\sum_{i=1}^{l} \lambda_i = n$. The integers λ_i are called the *parts* of λ, and the number of parts $l = l(\lambda)$ (counting multiplicities) is called the *length* of λ. Given two partitions $\lambda = \{\lambda_1, ..., \lambda_l\} \in P_n$ and $\mu = \{\mu_1, ..., \mu_k\} \in P_m$ with $m \leqslant n$, we write $\mu \leqslant \lambda$ if and only if $k \leqslant l$ and $\mu_i \leqslant \lambda_i$ for all $i \leqslant k$. Also, we define the set DP_n of *distinct partitions* of n to be the subset of P_n consisting of partitions with distinct parts. Thus $\lambda \in \mathrm{DP}_n$ if and only if $\lambda_i > \lambda_{i+1}$ for $1 \leqslant i < l$. The *parity* of λ is the parity of the permutation of an element in S_n with the same cycle type as λ. Thus λ is odd or even according to whether the corresponding element in S_n lies in $S_n \backslash A_n$ or in A_n. We also write $\mathrm{sgn}(\lambda) = -1$ if λ is odd and 1 if λ is even.

The irreducible faithful characters of $\hat{S}_n$ are indexed by the distinct partitions $\lambda = \{\lambda_1, ..., \lambda_l\}$ in DP_n. For each even partition in DP_n there is one irreducible character denoted ϕ_λ. For each odd partition, there are two irreducible characters, denoted ϕ_λ^+, ϕ_λ^-. In this second case, we use the notation ϕ_λ to denote either ϕ_λ^+ or ϕ_λ^-. The characters ϕ_λ are often called *spin characters*. The character ϕ_λ for $\lambda = \{n\}$ is often called the *basic spin character* because it is the restriction of the spin character of the orthogonal group $O_{n-1}(\mathbf{R})$. If $\hat{H}$ is any subgroup of $\hat{S}_n$ containing the center Z of $\hat{S}_n$, we write H for the corresponding subgroup $\hat{H}/Z \leqslant \hat{S}_n/Z = S_n$. If H is a subgroup of S_n, we write $\hat{H}$ for the full preimage of H in $\hat{S}_n$.

Before stating our main results, we need to introduce one more piece of notation. The group S_n contains *parabolic* subgroups $S_{n-a} \times S_a$, acting naturally on $\{1, ..., n-a\}$ and $\{n-a+1, ..., n\}$. As a convenience we write $S_{n-a,a} = S_{n-a} \times S_a$. Also, if $n = ab$, then S_n contains subgroups $S_a \wr S_b$, a wreath product of S_a by S_b of order $(a!)^b\, b!$; these act transitively but imprimitively on $\{1, ..., n\}$.

THEOREM 1.1. *Suppose ϕ_λ is a faithful irreducible character of $\hat{S}_n$, $n \geqslant 5$, and suppose $\hat{H}$ is a maximal subgroup of $\hat{S}_n$. Then $\phi_\lambda \mid \hat{H}$ is irreducible if and only if one of the following holds.*

(1) $H = A_n$, λ *odd.*

(2) $H = S_{n-1,1}$, $\lambda = \{l+r, l-1+r, \ldots, 1+r\}$, $n = \frac{1}{2}l(l+1) + rl$, $l \geq 2$ *and with* λ *odd and* $r \geq 0$, *or with* λ *even and* $r = 0$.

(3) $H = S_{n-2,2}$, $\lambda = \{l, l-1, \ldots, 1\}$, $n = \frac{1}{2}l(l+1)$.

(4) $H = S_{n-a,a}$, $\lambda = \{n\}$, n *even*, $a < \frac{1}{2}n$.

(5) $H = S_a \wr S_b$, $n = ab$, $a, b \geq 2$, $\lambda = \{n\}$.

(6) $H = S_5 \wr S_2$, $n = 10$, $\lambda = \{4, 3, 2, 1\}$.

(7) $H = S_2 \wr S_3$ *or* $S_3 \wr S_2$, $n = 6$, $\lambda = \{3, 2, 1\}$.

(8) $H = \mathbf{Z}_5 : \mathbf{Z}_4$, $n = 5$, $\lambda = \{5\}$ *or* $\{3, 2\}$.

(9) $H = S_5$, $n = 6$, $\lambda = \{6\}$ *or* $\{3, 2, 1\}$, H *transitive.*

(10) $H = \mathrm{Aut}(A_6)$, $n = 10$, $\lambda = \{10\}$.

(11) $H = M_{12}$, $n = 12$, $\lambda = \{12\}$.

Remark. Here $\mathrm{Aut}(A_6)$ is the full automorphism group of A_6 and $\mathbf{Z}_5 : \mathbf{Z}_4$ is the Frobenius group of order 20, which is the Sylow 5-normalizer in S_5. Moreover M_{12} denotes the Mathieu group of order $2^6 \cdot 3^3 \cdot 5 \cdot 11$.

To discuss the irreducible characters of $\hat{A}_n$, we use the fact that $\phi_\lambda^+ \mid \hat{A}_n = \phi_\lambda^- \mid \hat{A}_n$ is irreducible if λ is odd (see [18, 22]), and we denote this character of $\hat{A}_n$ by η_λ. If λ is even, $\phi_\lambda \mid \hat{A}_n$ is a sum of two irreducible characters denoted η_λ^1 and η_λ^2. Thus the irreducible characters of $\hat{A}_n$ are η_λ, λ odd, and η_λ^1, η_λ^2, λ even. When λ is even, we use the notation η_λ to denote either η_λ^1 or η_λ^2.

THEOREM 1.2. *Suppose* η_λ *is a faithful irreducible character of* $\hat{A}_n$, $n \geq 5$, *and that* $\hat{H}$ *is a maximal subgroup of* $\hat{A}_n$. *Then* $\eta_\lambda \mid \hat{H}$ *is irreducible if and only if one of the following holds.*

(1) $H = S_{n-1,1} \cap A_n$, $\lambda = \{l+r, l-1+r, \ldots, 1+r\}$, $n = \frac{1}{2}l(l+1) + rl$, $r \geq 1$, $l \geq 2$, λ *even.*

(2) $H = S_{n-1,1} \cap A_n$, $\lambda = \{l, l-1, \ldots, 1\}$, $n = \frac{1}{2}l(l+1)$.

(3) $H = S_{n-2,2} \cap A_n$, $\lambda = \{l, l-1, \ldots, 1\}$, $n = \frac{1}{2}l(l+1)$.

(4) $H = S_{n-a,a} \cap A_n$, $\lambda = \{n\}$, n *odd*, $a < \frac{1}{2}n$.

(5) $H = S_a \wr S_b \cap A_n$, $n = ab$, $a, b \geq 2$, $\lambda = \{n\}$.

(6) $H = S_5 \wr S_2 \cap A_{10}$, $n = 10$, $\lambda = \{4, 3, 2, 1\}$.

(7) $H = S_3 \wr S_2 \cap A_6$, $n = 6$, $\lambda = \{3, 2, 1\}$.

(8) $H = D_{10}$, $n = 5$, $\lambda = \{5\}$.

(9) $H = A_5$, $n = 6$, $\lambda = \{6\}$, H *transitive.*

(10) $H = L_2(7)$, $n = 7$, $\lambda = \{7\}$.

(11) $H = AGL_3(2)$, $n = 8$, $\lambda = \{8\}$ *or* $\{7, 1\}$.

(12) $H = AGL_2(3) \cap A_9$, $n = 9$, $\lambda = \{9\}$.

(13) $H = \mathrm{Aut}(L_2(8))$, $n = 9$, $\lambda = \{9\}$.

(14) $H = M_{10}$, $n = 10$, $\lambda = \{10\}$.

(15) $H = M_{11}$, $n = 11$, $\lambda = \{11\}$.

(16) $H = M_{12}$, $n = 12$, $\lambda = \{12\}$ *or* $\{11, 1\}$.

Remark. Here $AGL_d(p)$ denotes the split extension of an elementary abelian group of order p^d (p prime) by $GL_d(p)$, which has a natural permutation representation of degree p^d. The groups M_{11} and M_{12} are Mathieu groups, and M_{10} denotes a subgroup of index 11 in M_{11} (in fact M_{10} contains A_6 as a subgroup of index 2). In cases (10)–(16) there are some subtleties concerning the number of conjugacy classes of subgroups and the number of characters which remain irreducible. The interested reader will find details in Section 6.

THEOREM 1.3. *Suppose η_λ is a faithful irreducible character of $\hat{A}_n$, $n \geq 5$, and that $\hat{H}$ is a quasisimple proper subgroup of $\hat{A}_n$. Then $\eta_\lambda \mid \hat{H}$ is irreducible if and only if one of the following holds.*

(1) $H = A_{n-1}$ *as in* (1) *or* (2) *of Theorem* 1.2, *with* $n \geq 6$.

(2) $H = A_{n-2}$, $\lambda = \{l, l-1, ..., ..., 1\}$, $n = \frac{1}{2}l(l+1)$, $n \geq 10$, λ *even*.

(3) $H = A_{n-1}$, $\lambda = \{n\}$, n *odd*, $n \geq 7$.

(4) $H = A_5$, $n = 6$, $\lambda = \{6\}$, H *transitive*.

(5) $H = L_2(7)$, $n = 7$, $\lambda = \{7\}$.

(6) $H = L_2(8)$, $n = 9$, $\lambda = \{9\}$.

(7) $H = M_{11}$, $n = 11$, $\lambda = \{11\}$.

(8) $H = M_{12}$, $n = 12$, $\lambda = \{12\}$ *or* $\{11, 1\}$.

(9) $H = A_5$, $n = 7$, $\lambda = \{7\}$, *H has orbits with sizes* 1 *and* 6.

(10) $H = M_{12}$, $n = 13$, $\lambda = \{13\}$, *H has orbits with sizes* 1 *and* 12.

Remark. The occurrence of a triple of quasisimple groups $K < H < G$ all acting absolutely irreducibly on a vector space is quite rare. Thus it is worthwhile to highlight these occurrences in Theorem 1.3. When n and λ satisfy the conditions in part (2) of Theorem 1.3, η_λ stays irreducible down the chain $\hat{A}_{n-2} \leq \hat{A}_{n-1} \leq \hat{A}_n$. Similarly, parts (9) and (10) show that $\eta_{\{7\}}$ and $\eta_{\{13\}}$ stay irreducible down the chains $\hat{A}_5 \leq \hat{A}_6 \leq \hat{A}_7$ and $\hat{M}_{12} \leq \hat{A}_{12} \leq \hat{A}_{13}$, respectively (here $\hat{M}_{12}$ denotes the double cover of M_{12}).

The multiplier of A_n is $\mathbf{Z}_2$, except when $n = 6$ or 7, in which case the multiplier is $\mathbf{Z}_6$ [18]. Thus to give complete information on the restrictions of projective characters of the alternating groups, we must determine when faithful characters of the triple covers and the six-fold covers of A_6 and A_7 remain irreducible. This information is recorded in Theorem 1.4.

THEOREM 1.4. *Assume that $n = 6$ or 7 and write $\tilde{A}_n$ and $\bar{A}_n$ for the three-fold and six-fold covers of A_n, respectively. Also assume that H is a maximal subgroup of A_n, and let $\tilde{H}$ and $\bar{H}$ denote the full preimages of H in $\tilde{A}_n$ and $\bar{A}_n$, respectively.*

(i) *The character χ of $\tilde{A}_n$ remains irreducible upon restriction to $\tilde{H}$ if and only if one of the following holds.*

(1) $n = 6$ *and* $\chi(1) = 3$.

(2) $H = A_6$, $n = 7$, $\chi(1) = 6$.

(3) $H = A_6$, $n = 7$, $\chi(1) = 15$, χ *takes the value* -1 *on involutions.*

(4) $H = L_2(7)$, $n = 7$, $\chi(1) = 6$.

(ii) *The character χ of $\bar{A}_n$ remains irreducible upon restriction to $\bar{H}$ if and only if one of the following holds.*

(5) $H = A_5$, $n = 6$, $\chi(1) = 6$, *H either transitive or intransitive.*

(6) $n = 7$ *and* $\chi(1) = 6$.

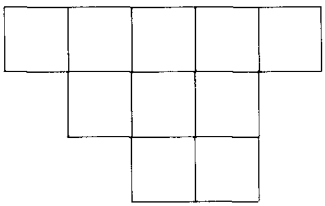
Remark. The group $\tilde{A}_7$ has four characters of degree 15, coming in two pairs of complex conjugates. One pair assumes the value -1 on (the unique class of) involutions, while the other assumes the value 3. Only those in the former pair remain irreducible upon restriction to $\tilde{A}_6$ (see part (3) in Theorem 1.4). Observe that $\tilde{A}_6$ and $\bar{A}_7$ have the unusual property that the faithful characters of degrees 3 and 6, respectively, remain irreducible upon restriction to any maximal subgroup (see parts (1) and (6)).

The theorems are proved using techniques from [17] and results in [22] which describe how the characters ϕ_λ restrict to certain subgroups. The results of [22] are described in Section 2 and some easy consequences are drawn. A lower bound for the degree of ϕ_λ is obtained in Section 3. In Section 4, information about the decomposition of $\phi_\lambda \bar{\phi}_\lambda$ is used to show that H acts primitively, except for certain λ. In Section 5 we use the techniques in [17] to reduce the analysis to one of four specific situations. The classification of finite simple groups is then used to describe primitive subgroups of S_n of large order in Section 6. These results are used in Sections 7 and 8 to prove Theorem 1.1. Theorem 1.2 and Theorem 1.3 are proved in Sections 9 and 10, respectively, using the earlier results and techniques. We omit the proof of Theorem 1.4. This is essentially an easy exercise once one has a list of the maximal subgroups of A_6 and A_7 and the relevant character tables. All this information appears in [5].

2. Background Results

In this section we summarize information about the irreducible characters ϕ_λ of $\hat{S}_n$. Recent papers [22, 16, 24] provide information we need about the decomposition of $\phi_\lambda \bar{\phi}_\lambda$. Here $\lambda \in \mathrm{DP}_n$, the set of partitions of n into distinct parts. As mentioned above, the characters ϕ_λ were first defined by Schur in [18]. Other papers describing aspects of these characters and their properties are [7, 13, 14].

It is natural to consider the *shifted diagram* for a partition λ in DP_n. This is the usual diagram for $\lambda = \{\lambda_1, ..., \lambda_l\}$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_l$, but with row j shifted $j-1$ columns to the right. Thus the shifted diagram for $\{5, 3, 2\}$ is



A. If ϕ is any character of $\hat{S}_n$, the product $(\mathrm{sgn})\phi$ is also a character, where sgn denotes the sign character of S_n of degree 1. The characters ϕ and $(\mathrm{sgn})\phi$ are said to be *associated.* In the case where $\phi = (\mathrm{sgn})\phi$, we say that ϕ is *self-associated.* If ϕ is irreducible, then the following are equivalent: ϕ is self-associated, ϕ vanishes on $\hat{S}_n \backslash \hat{A}_n$, and the restriction $\phi \mid \hat{A}_n$ is a sum of two distinct irreducible characters. Moreover, ϕ_λ is self-associated if and only if λ is even. Thus when λ is odd, there are two associated characters which we call ϕ_λ^+ and ϕ_λ^-. The notation ϕ_λ denotes the faithful irreducible character of $\hat{S}_n$ indexed by λ if λ is even, or either of $\phi_\lambda^\pm$ if λ is odd. These are all the faithful characters of $\hat{S}_n$ [18]. If λ is odd, then ϕ_λ^+ and ϕ_λ^- agree on all classes of $\hat{S}_n$, except those whose image in S_n has cycle type λ. Thus if λ is odd and $y \in \hat{S}_n \backslash \hat{A}_n$, then $\phi_\lambda^\pm(y) = 0$, unless the image of y in S_n has cycle type λ, in which case $\phi_\lambda^+(y) = -\phi_\lambda^-(y)$. If $y \in \hat{A}_n$, then $\phi_\lambda^+(y) = \phi_\lambda^-(y)$.

B. Using part A, if λ is odd, $\phi_\lambda \mid \hat{A}_n = \eta_\lambda$ is irreducible. If λ is even, $\phi_\lambda \mid \hat{A}_n = \eta_\lambda^1 + \eta_\lambda^2$, with η_λ^1 and η_λ^2 irreducible of the same degree [18, 22]. We use the notation η_λ to denote the irreducible character $\phi_\lambda \mid \hat{A}_n$ if λ is odd, and either of the irreducibles η_λ^1 or η_λ^2 if λ is even. Any irreducible character of $\hat{A}_n$ is one of these η_λ.

C. The degree of ϕ_λ is given by

$$\phi_\lambda(1) = \frac{2^{(n-l)/2} n!}{\varepsilon_\lambda \prod_{i=1}^{l} \lambda_i!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \tag{2.1}$$

where $\lambda=\{\lambda_1,\ldots,\lambda_l\}$ and

$$\varepsilon_\lambda=\begin{cases}\sqrt{2} & \text{if } \lambda \text{ is odd}\\ 1 & \text{if } \lambda \text{ is even.}\end{cases} \tag{2.2}$$

This definition of ε_λ applies regardless of whether λ has distinct parts. Note that λ is even if and only if $n-l$ is even, and so $2^{(n-l)/2}/\varepsilon_\lambda$ is always an integer. Indeed, its value is $2^{\lfloor (n-l)/2\rfloor}$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. Moreover

$$\phi_{\{n\}}(1)=\begin{cases}2^{(n-2)/2} & \text{if } n \text{ is even}\\ 2^{(n-1)/2} & \text{if } n \text{ is odd.}\end{cases} \tag{2.3}$$

D. Let $\mu=\{\mu_1,\ldots,\mu_r\}$ be any partition of n. Denote by $S_\mu=S_{\mu_1,\ldots,\mu_r}$ the subgroup $S_{\mu_1}\times\cdots\times S_{\mu_r}$ acting naturally on any subdivision of $I=\{1,\ldots,n\}$ into sets of sizes $\mu_1,\ldots,\mu_r$. Let $\hat{S}_\mu=\hat{S}_{\mu_1,\ldots,\mu_r}$ be the full inverse image of S_μ in $\hat{S}_n$. The order of $\hat{S}_\mu$ is then $2\prod_{i=1}^r(\mu_i!)$.

Certain natural irreducible characters $\phi_{\beta_1,\ldots,\beta_r}$ of $\hat{S}_\mu$ are defined in [22, Sects. 4.2 and 4.3], where $\mu=\{\mu_1,\ldots,\mu_r\}\in P_n$ and $\beta_i\in \mathrm{DP}_{\mu_i}$. An alternate treatment appears in Section 8. We have

$$\phi_{\beta_1,\ldots,\beta_r}(1)=2^{\lfloor s/2\rfloor}\prod_{i=1}^{r}\phi_{\beta_i}(1), \tag{2.4}$$

where ϕ_{β_i} is a faithful irreducible character of $\hat{S}_{\mu_i}$ corresponding to β_i, and s is the number of ϕ_{β_i} which are not self-associated (recall, ϕ_{β_i} is self-associated if and only if β_i is an even partition of μ_i). Furthermore,

$$\phi_{\beta_1,\ldots,\beta_r}(\pi_1\cdots\pi_r)=2^{\lfloor s/2\rfloor}\prod_{i=1}^{r}\phi_{\beta_i}(\pi_i), \tag{2.5}$$

provided all $\pi_i\in S_{\mu_i}$ are even. Its value for other $\pi_1\cdots\pi_r$, $\pi_i\in S_{\mu_i}$, is described in [22, Sect. 4.2], and is usually 0. These characters are all faithful irreducible characters of $\hat{S}_\mu$, and any such character is of this form. If s is even, there is one character, and if s is odd, there are two characters $\phi^{\pm}_{\beta_1,\ldots,\beta_r}$. As usual, we use the notation $\phi_{\beta_1,\ldots,\beta_r}$ to denote either of these characters when s is odd.

E. Assume $1\leqslant a\leqslant n-1$, and let $\mu\in \mathrm{DP}_{n-a}$, $\nu\in \mathrm{DP}_a$, and $\lambda\in \mathrm{DP}_n$. The results of [22, Theorem 8.1] give the multiplicity of $\phi_{\mu,\nu}$ as a constituent of $\phi_\lambda\mid\hat{S}_{n-a,a}$. It is

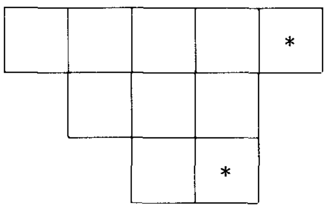
$$\frac{1}{\varepsilon_\lambda\varepsilon_{\mu\cup\nu}}\,2^{(l(\mu)+l(\nu)-l(\lambda))/2}f^\lambda_{\mu\nu}, \tag{2.6}$$

where $f^\lambda_{\mu\nu}$ is an integer appearing as a multiplication constant for an algebra of symmetric functions. Sometimes we write $f_{\mu\nu}=f^\lambda_{\mu\nu}$ when λ is clear from the context. There is an exception when λ is odd and $\lambda=\mu\cup\nu$; here there are two characters $\phi^{\pm}_{\mu,\nu}$, and the multiplicity is 0 for one and 1 for the other. This is referred to as the *exceptional case.*

The coefficients $f^\lambda_{\mu\nu}$ can be computed according to a rule similar in spirit to the Littlewood–Richardson Rule in terms of fillings of λ/μ, where λ/μ is the shifted diagram of λ with the shifted diagram of μ removed. In particular, $f^\lambda_{\mu\nu}$ is 0 unless $\mu \leqslant \lambda$ and $\nu \leqslant \lambda$. The details are complicated and for the most part unnecessary for our work. When needed, we mention the diagram λ/μ and the fillings which are appropriate and leave the details to the reader.

One special case has been by handled by several authors and gives a branching rule from $\hat{S}_n$ to $\hat{S}_{n-1,1}$ [7, 13, 14]. In this case, $\nu = 1$, and so the characters occurring as constituents of $\phi_\lambda \mid \hat{S}_{n-1,1}$ are among $\phi_{\mu,\{1\}}$, where $\mu \in \mathrm{DP}_{n-1}$. Clearly $\hat{S}_{n-1,1} \cong \hat{S}_{n-1}$ and $\phi_{\mu,\{1\}}$ becomes ϕ_μ. We make such identifications throughout.

Suppose $\lambda \in \mathrm{DP}_n$. The partitions $\mu \in \mathrm{DP}_{n-1}$ for which $\mu \leqslant \lambda$ are the partitions obtained by removing a corner from the right-hand side of the shifted diagram for λ. For example, if $\lambda = \{5, 3, 2\}$ as above,



There are corners in the $(1, 5)$ and $(3, 2)$ locations in this "shifted" notation. Thus the possible μ are $\{4, 3, 2\}$ and $\{5, 3, 1\}$; the possible λ/μ are the corners $(1, 5)$ and $(3, 2)$. The characters occurring with non-zero multiplicity in $\phi_{\{5,3,2\}} \mid \hat{S}_9$ are $\phi_{\{4,3,2\}}$ and $\phi_{\{5,3,1\}}$; their multiplicity is 1.

Suppose $\lambda \in \mathrm{DP}_n$, $\mu \in \mathrm{DP}_{n-1}$, and $\mu \leqslant \lambda$. Then λ/μ is a right-hand corner of the shifted diagram for λ. Suppose λ/μ is not in the position $(l, 1)$ when $\lambda_l = 1$. Then $l(\mu) = l(\nu)$, exactly one of λ, $\mu \cup \{1\}$ is odd, and $l(\mu) + l(1) - l(\lambda) = 1$. Consequently

$$\frac{1}{\varepsilon_\lambda \varepsilon_{\mu\cup\{1\}}} 2^{(l(\mu)+l(1)-l(\lambda))/2} f^\lambda_{\mu,1} = f^\lambda_{\mu,1}. \tag{2.7}$$

There is exactly one filling with 1, 1′ of this box, namely, 1, satisfying the conditions in [22, Theorem 8.3]. The only condition here being the first (and so only), entry is a 1. If $\lambda_l = 1$ and λ/μ is $(l, 1)$, then $l(\mu) + l(1) - l(\lambda) = 0$ and $\varepsilon_\lambda = \varepsilon_{\mu\cup\{1\}}$. Again $f^\lambda_{\mu,1} = 1$. If λ is even, then the multiplicity of $\phi_{\lambda/\mu}$ in $\phi_\lambda \mid \hat{S}_{n-1}$ is 1. If λ is odd, then there are two possible characters $\phi^\pm_{\lambda/\mu}$ as λ/μ is also odd. This is the exceptional case referred to above. Only one of these associates appears as a constituent. We have shown that if $\lambda \in \mathrm{DP}_n$,

$$\phi_\lambda \mid \hat{S}_{n-1} = \sum \phi_{\lambda/c}, \tag{2.8}$$

where the summation is over the corners of the shifted diagram for λ, and if λ/c is odd, both associates occur, except when λ is odd, $\lambda_l = 1$, and c is the bottom corner, in which case only one associate occurs.

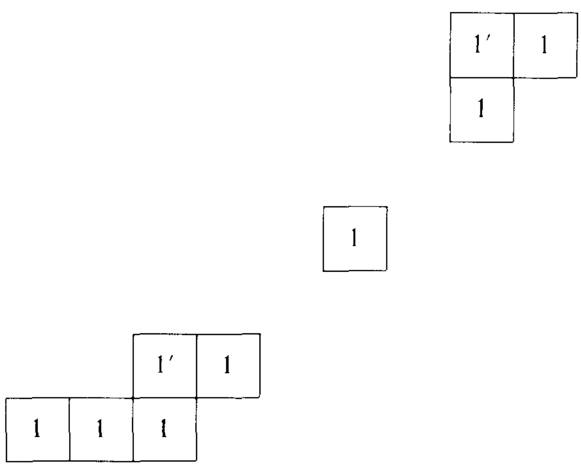
F. In several situations we wish to know whether $\phi_\lambda \mid \hat{S}_{n-a,a}$ has more than one constituent. Each such constituent is of the form $\phi_{\mu,\nu}$, where $\mu \in \mathrm{DP}_{n-a}$ and $\nu \in \mathrm{DP}_{n-a}$. If $\phi_{\mu,\nu}$ is a constituent, then $\mu \leqslant \lambda$ and $\nu \leqslant \lambda$. The following discussion shows that whenever $\mu \leqslant \lambda$, there is some ν for which $\phi_{\mu,\nu}$ is a constituent of $\phi_\lambda \mid \hat{S}_{n-a,a}$.

PROPOSITION 2.1. *Suppose $\lambda \in \mathrm{DP}_n$, $\mu \in \mathrm{DP}_{n-a}$, and $\mu \leqslant \lambda$. Then there is a partition $\nu \in \mathrm{DP}_a$ such that $f^{\lambda}_{\mu,\nu} \neq 0$. In particular, $\langle \phi_\lambda \mid \hat{S}_{n-a,a}, \phi_{\mu,\nu} \rangle \neq 0$.*

Proof. Arrange λ as a shifted diagram and place the shifted diagram for μ in the upper-left portion of λ. It suffices to find a filling of λ/μ for which the word $w = w(s)$ described in [22, Theorem 8.3] satisfies conditions (i) and (ii) of that theorem.

A set of distinct nodes in the shifted diagram for λ is said to be a *descending strip* if the nodes can be labelled $(i_1, j_1), \ldots, (i_r, j_r)$, having the property $i_{k+1} \geqslant i_k$ and $j_{k+1} \leqslant j_k$ for $1 \leqslant k \leqslant r-1$. Thus a typical descending strip appears below.

Let γ be a descending strip. A node of γ is said to be *vertical* if there exists a node in γ directly below it—that is, (i, j) is vertical if and only if $(i+1, j)$ also lies in γ. The nodes which are not vertical are called *horizontal.* Furthermore, the *standard i-filling* of γ is obtained by labelling all horizontal nodes i and all vertical nodes i'. For example,



The standard i-filling has i at the lower left, has at most one i' in any row and at most one i in any column, and is non-decreasing along rows and columns, subject to the relation $i' < i$.

We now consider the diagram λ/μ. The set of nodes in λ/μ which are adjacent to μ forms a descending strip γ_1. We impose the standard 1-filling on γ_1. Now the set of nodes in $\lambda/\mu/\gamma_1$ which are adjacent to γ_1 forms a descending strip γ_2, on which we impose the standard 2-filling. Continue in this way until λ/μ has been filled.

We leave it to the reader to check that the conditions of [22, Theorem 8.3] are satisfied. ∎

COROLLARY 2.2. *If* $3 \leqslant a \leqslant n-3$ *and* $\lambda \neq \{n\}$, *then there are at least two distinct* μ, μ' *in* DP_{n-a} *with* $\mu, \mu' \leqslant \lambda$, *Therefore* $\phi_\lambda \mid \hat{S}_{n-a,a}$ *has constituents of the form* $\phi_{\mu,\nu}$, $\phi_{\mu',\nu'}$ *for suitable* ν, ν', *and so* $\phi_\lambda \mid \hat{S}_{n-a,a}$ *has at least two distinct constituents* $\phi_{\mu,\nu}$, $\phi_{\mu',\nu'}$, *with* $\mu \neq \mu'$.

Proof. Remove a locations from the shifted diagram for λ starting from the bottom row and removing from the right to the left. This leaves μ in DP_{n-a} in shifted form. To determine μ', remove a locations in λ from the rightmost column and from the bottom to the top. As $n-3 \geqslant a \geqslant 3$ and $\lambda \neq \{n\}$, the partitions μ, μ' are distinct. Now apply Proposition 2.1. ∎

Remark. If $a=2$, then $\lambda = \{l, l-1, ..., 1\}$ would have to be excluded, and if $a=1$, then $\lambda = \{l+r, l-1+r, ..., 1+r\}$ would have to be excluded.

G. Here we record an elementary lemma in character theory which is used implicitly in several arguments below.

LEMMA 2.3. *Assume that* K *is a group with a subgroup* H *of index* 2, *and that* χ *is an irreducible character of* K. *Then* $\chi \mid H$ *is reducible if and only if* χ *vanishes on* $K \setminus H$. *When this occurs*, $\chi \mid H = \psi_1 + \psi_2$, *with* ψ_i *irreducible*, $\psi_1 \neq \psi_2$, *and if* $x \in K \setminus H$, *then* $\psi_1^x = \psi_2$, *where* $\psi_1^x(h) = \psi_1(xhx^{-1})$.

H. Results connected to Bertrand's Postulate are used in Sections 7 and 9 to show that there are relatively large primes dividing the degrees of certain irreducible characters of $\hat{S}_n$ and $\hat{A}_n$. This information is then used to show that these characters cannot remain irreducible upon restriction to particular subgroups whose orders are not divisible by these primes. A version of Bertrand's Postulate suitable for our needs was proved by Schur.

THEOREM 2.4 (Schur [19, Sect. 2]). *If* $n \geqslant 29$, *then there is a prime* p *satisfying* $n \leqslant p \leqslant \frac{5}{4}n$.

See also Gatteschi [8], where $\frac{5}{4}$ is replaced by $\frac{11}{9}$, and [10] for a review of this work. Another source of reference is "Mathematical Reviews," Vol. 9, p. 332, 1948.

I. We conclude Section 2 by stating a theorem due to Wagner, which serves as a convenience in some of the arguments in Section 6.

THEOREM 2.5 (Wagner [23]). *Assume that* $n \geqslant 10$, *and write* $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, *where* $0 \leqslant a_1 < a_2 < \cdots < a_s$. *If* $\lambda \in \mathrm{DP}_n$, *then* $\phi_\lambda(1)$ *is divisible by* $2^{\lfloor (n-s)/2 \rfloor}$ *and* $\eta_\lambda(1)$ *is divisible by* $2^{\lfloor (n-s-1)/2 \rfloor}$.

3. THE SMALLEST DEGREE

The branching rule (2.8) provides an easy proof that the basic spin representations have the smallest degrees of any faithful representation of $\hat{S}_n$, and a bit more.

THEOREM 3.1. *Suppose* $n \geqslant 4$. *Then*

$$\phi_\lambda(1) \geqslant \begin{cases} 2^{(n-2)/2} & \text{if } \lambda \text{ is odd} \\ 2^{(n-1)/2} & \text{if } \lambda \text{ is even.} \end{cases}$$

These are the degrees of $\phi_{\{n\}}$.

Proof. This is easily checked for $n = 4$. Assume therefore that $n \geqslant 5$. Let x be the number of right-side corners of the shifted diagram for λ and suppose $x \geqslant 2$. Then

$$\phi_\lambda \mid \hat{S}_{n-1} = \phi_{\lambda/c_1} + \phi_{\lambda/c_2} + \cdots,$$

where c_1, c_2 are distinct corners. So by induction,

$$\phi_\lambda(1) \geqslant \phi_{\lambda/c_1}(1) + \phi_{\lambda/c_2}(1) \geqslant 2^{(n-1-2)/2} + 2^{(n-1-2)/2} = 2^{(n-1)/2}.$$

Suppose now that $x = 1$, so there is just one corner c. If $l = 1$, then $\lambda = \{n\}$, and equality holds. So assume $l \geqslant 2$. If λ is odd and $\lambda_l \neq 1$, then $\phi_\lambda \mid \hat{S}_{n-1}$ has a constituent $\phi_{\lambda/c}$. Moreover, λ/c is even, and so by induction its degree is at least $2^{(n-1-1)/2}$, and the result follows. If λ is even and $\lambda_l \neq 1$, then λ/c is odd and so there are two constituents $\phi^{\pm}_{\lambda/c}$ in $\phi_\lambda \mid \hat{S}_{n-1}$. Here

$$\phi_\lambda(1) \geqslant 2 \times 2^{(n-1-2)/2} = 2^{(n-1)/2}.$$

This leaves $\lambda_l = 1$. Here $\lambda = \{l, l-1, ..., 1\}$. If $\tilde{\lambda} = \{l-1, l-2, ..., 1\} \in \mathrm{DP}_{n-l}$, an easy check using (2.1) shows

$$\phi_\lambda(1) = \frac{2^{(l-1)/2} \varepsilon_{\tilde{\lambda}} n(n-1) \cdots (n-l+1)\, \phi_{\tilde{\lambda}}(1)}{\varepsilon_\lambda (2l-1)(2l-2) \cdots l}.$$

Now using induction and checking small values shows $\phi_\lambda(1) \geqslant 2^{(n-1)/2}$. ∎

COROLLARY 3.2. *Any faithful irreducible character of $\hat{A}_n$ has degree at least $2^{(n-3)/2}$.*

Proof. This is clear from Theorem 3.1 and part B of Section 2. ∎

4. ANALYSIS OF $\phi_\lambda \bar{\phi}_\lambda$

In this section, let ϕ_λ be a faithful character of $\hat{S}_n$ with $n \geqslant 4$. This means $\phi_\lambda \bar{\phi}_\lambda$ is a character of S_n. Suppose the irreducible characters of S_n are denoted ψ_μ, where $\mu \in P_n$ and ψ_μ is the character of the Specht module indexed by μ (see [9]). Let

$$\phi_\lambda \bar{\phi}_\lambda = \psi_{\{n\}} + s\psi_{\{n-1,1\}} + t\psi_{\{n-2,2\}} + \sum d_\mu \psi_\mu, \tag{4.1}$$

where the final summation is over partitions in P_n not equal to $\{n\}$, $\{n-1,1\}$, or $\{n-2,2\}$. As ϕ_λ is irreducible and $\psi_{\{n\}}$ is the trivial character, we have $\langle \psi_{\{n\}}, \phi_\lambda \bar{\phi}_\lambda \rangle_{S_n} = 1$.

In order to apply the techniques from [17], we need to know when s and t are zero. These multiplicities can be determined using the branching rules for restrictions to large subgroups. In particular, suppose $\phi_\lambda | \hat{S}_{n-1,1}$ has norm γ. Then

$$\begin{aligned}
\gamma &= \langle \phi_\lambda | \hat{S}_{n-1,1}, \phi_\lambda | \hat{S}_{n-1,1} \rangle_{\hat{S}_{n-1,1}} \\
&= \langle \phi_\lambda \bar{\phi}_\lambda | S_{n-1,1}, 1 \rangle_{S_{n-1,1}} \\
&= \langle \psi_{\{n\}} + s\psi_{\{n-1,1\}} + t\psi_{\{n-2,2\}} \\
&\qquad + \sum d_\mu \psi_\mu | S_{n-1,1}, 1 \rangle_{S_{n-1,1}} \\
&= 1 + s.
\end{aligned} \tag{4.2}$$

This follows because $\langle \psi_{\{n\}} | S_{n-1,1}, 1 \rangle_{S_{n-1,1}} = \langle \psi_{\{n-1,1\}} | S_{n-1,1}, 1 \rangle_{S_{n-1,1}} = 1$ and $\langle \psi_\mu | S_{n-1,1}, 1 \rangle_{S_{n-1,1}} = 0$ for $\mu \neq \{n\}$, $\{n-1,1\}$, by Frobenius Reciprocity. Recall that the permutation character on the cosets of $S_{n-1,1}$ is $\psi_{\{n\}} + \psi_{\{n-1,1\}}$ and that on the cosets of $S_{n-2,2}$ is $\psi_{\{n\}} + \psi_{\{n-1,1\}} + \psi_{\{n-2,2\}}$. This means

$$\begin{aligned}
\langle \psi_{\{n\}} | S_{n-2,2}, 1 \rangle_{S_{n-2,2}} &= \langle \psi_{\{n-1,1\}} | S_{n-2,2}, 1 \rangle_{S_{n-2,2}} \\
&= \langle \psi_{\{n-2,2\}} | S_{n-2,2}, 1 \rangle_{S_{n-2,2}} = 1,
\end{aligned}$$

and $\langle \psi_\mu | S_{n-2,2}, 1 \rangle_{S_{n-2,2}} = 0$ for $\mu \neq \{n\}$, $\{n-1, 1\}$, $\{n-2, 2\}$. If δ is the norm of $\phi_\lambda | \hat{S}_{n-2,2}$, then

$$\begin{aligned} \delta &= \langle \phi_\lambda \bar{\phi}_\lambda | S_{n-2,2}, 1 \rangle_{S_{n-2,2}} \\ &= 1 + s + t. \end{aligned} \tag{4.3}$$

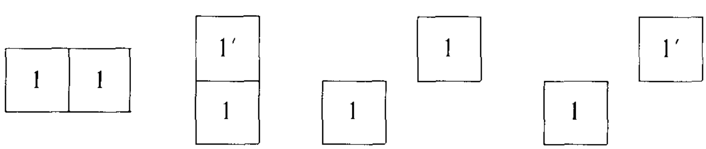
These arguments prove the following proposition.

PROPOSITION 4.1. *Suppose ϕ_λ is an irreducible faithful character of $\hat{S}_n$, $n \geq 4$, and $\phi_\lambda \bar{\phi}_\lambda = \psi_{\{n\}} + s\psi_{\{n-1,1\}} + t\psi_{\{n-2,2\}} + \sum d_\mu \psi_\mu$, as in (4.1). Let $\gamma = \langle \phi_\lambda | \hat{S}_{n-1,1}, \phi_\lambda | \hat{S}_{n-1,1} \rangle$, $\delta = \langle \phi_\lambda | \hat{S}_{n-2,2}, \phi_\lambda | \hat{S}_{n-2,2} \rangle$. Then $\gamma = s + 1$ and $\delta = 1 + s + t$. In particular, $t \neq 0$ if and only if $\phi_\lambda | \hat{S}_{n-2,2}$ has greater norm than $\phi_\lambda | \hat{S}_{n-1,1}$.*

Proposition 4.1 shows that s and t can be calculated in terms of the norms of $\phi_\lambda | \hat{S}_{n-1,1}$ and $\phi_\lambda | \hat{S}_{n-2,2}$. We have already seen the branching rule from $\hat{S}_n$ to $\hat{S}_{n-1,1}$ in (2.8), and the following discussion yields a proof of Lemma 4.2, which provides a branching rule from $\hat{S}_n$ to $\hat{S}_{n-2,2}$.

There are two faithful characters of $\hat{S}_2$ indexed by $\{2\}$. Note that $\hat{S}_2$ is cyclic of order 4 and $\phi^{\pm}_{\{2\}}(1) = 1$. The constituents of $\phi_\lambda | \hat{S}_{n-2,2}$ are characters $\phi_{\mu,\{2\}}$, where the shifted diagram for μ is obtained by first removing a corner c_1 from λ and then removing a corner c_2 from the shifted $(n-1)$-diagram λ/c_1. In this way, $\mu = \lambda/c_1/c_2$ is a shifted $(n-2)$-diagram. If c_1 and c_2 are in different rows and columns, call λ/μ a *disjoint pair*. If c_1 and c_2 are in the same row, call λ/μ a *horizontal pair*, and if they are in the same column, call λ/μ a *vertical pair*.

We need to know the values of the constants $f^\lambda_{\mu,\{2\}} = f_{\mu,\{2\}}$ connected to these μ to apply the results of [22]. The values of $f_{\mu,\{2\}}$ are 1 for horizontal or vertical pairs and 2 for disjoint pairs. The fillings are as follows:



$f_{\mu,\{2\}} = 1$ $\qquad$ $f_{\mu,\{2\}} = 1$ $\qquad$ $f_{\mu,\{2\}} = 2.$ $\qquad$ (4.4)

The results of [22, Theorem 8.1] now give the multiplicity of $\phi_{\mu,\{2\}}$ as a constituent of $\phi_\lambda | \hat{S}_{n-2,2}$ using Frobenius Reciprocity. This multiplicity is

$$\frac{1}{\varepsilon_\lambda \varepsilon_{\mu \cup \{2\}}} 2^{(l(\mu)+1-l(\lambda))/2} f_{\mu,\{2\}}, \tag{4.5}$$

unless λ is odd and $\lambda = \mu \cup \{2\}$. In this case there are two associates, one of which occurs as a constituent with multiplicity 1 (see (2.6)).

The coefficients can be evaluated, but they depend on whether λ/μ contains the bottom row of λ. Suppose it does not. Then $l(\mu) = l(\lambda)$ and so $l(\mu) + 1 - l(\lambda) = 1$. Also $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\mu)$ and $\operatorname{sgn}(\mu \cup \{2\}) = -\operatorname{sgn}(\mu)$. This means $2^{(l(\mu)+1-l(\lambda))/2}/\varepsilon_\lambda \varepsilon_{\mu \cup \{2\}} = 1$. If λ/μ does contain the bottom row of λ, then $l(\mu) + 1 - l(\lambda) = 0$. Also $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\mu \cup \{2\})$. This means $\varepsilon_\lambda \varepsilon_{\mu \cup \{2\}} = 1$ if λ is even and 2 if λ is odd. This shows that

$$\frac{2^{(l(\mu)+1-l(\lambda))/2}}{\varepsilon_\lambda \varepsilon_{\mu \cup \{2\}}} = \begin{cases} 1 & \text{if } \lambda \text{ is even} \\ 1 & \text{if } \lambda/\mu \text{ does not contain the bottom row} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{4.6}$$

We note from part D of Section 2 (2D) that if μ is odd, there is just one character of the form $\phi_{\mu,\{2\}}$ of degree $2 \times \phi_\mu(1)$. If μ is even, there are two characters, $\phi^{\pm}_{\mu,\{2\}}$, each of degree $\phi_\mu(1)$. By using the information in the analysis above, along with the results in 2D and 2E, we can now display the branching rule for $\phi_\lambda \mid \hat{S}_{n-2,2}$. In order to account for the different positions of the bottom corner of λ/μ, we divide the sum into several parts. Denote the bottom corner of λ/μ by c_b, and suppose c_λ is the corner (l, λ_l).

LEMMA 4.2. *Using the notation and terminology of the preceding discussion, we have the following.*

(i) *If λ is even, then*

$$\begin{aligned}
\phi_\lambda \mid \hat{S}_{n-2,2} = & \sum_{\substack{\lambda/\mu \text{ a disjoint pair,} \\ \text{excluding } \lambda_l = 1 \text{ and } c_b = c_\lambda}} 2(\phi^+_{\mu,\{2\}} + \phi^-_{\mu,\{2\}}) \\
& + \sum_{\substack{\lambda/\mu \text{ a disjoint pair,} \\ \lambda_l = 1,\, c_b = c_\lambda}} 2\phi_{\mu,\{2\}} \\
& + \sum_{\substack{\lambda/\mu \text{ a vertical pair,} \\ \text{excluding } \lambda_l = 1 \text{ and } c_b = c_\lambda}} (\phi^+_{\mu,\{2\}} + \phi_{\mu,\{2\}}) \\
& + \underset{\substack{\lambda/\mu \text{ a vertical pair,} \\ \lambda_l = 1,\, c_b = c_\lambda,\, \lambda_{l-1} = 2}}{\phi_{\mu,\{2\}}} \\
& + \sum_{\substack{\lambda/\mu \text{ a horizontal pair,} \\ \text{excluding } \lambda_l = 2 \text{ and } \lambda/\mu \text{ the bottom row}}} (\phi^+_{\mu,\{2\}} + \phi^-_{\mu,\{2\}}) \\
& + \underset{\substack{\lambda/\mu \text{ a horizontal pair,} \\ \lambda_l = 2,\, \mu \text{ the bottom row}}}{\phi_{\mu,\{2\}}} .
\end{aligned}$$

(ii) *If λ is odd, then*

$$\begin{aligned}
\phi_\lambda \mid \hat{S}_{n-2,2} = & \sum_{\substack{\lambda/\mu \text{ a disjoint pair,} \\ \text{excluding } \lambda_l = 1 \text{ and } c_b = c_\lambda}} 2\phi_{\mu,\{2\}} \\
& + \sum_{\substack{\lambda/\mu \text{ a disjoint pair,} \\ \lambda_l = 1,\, c_b = c_\lambda}} (\phi^{+}_{\mu,\{2\}} + \phi^{-}_{\mu,\{2\}}) \\
& + \sum_{\substack{\lambda/\mu \text{ a vertical pair,} \\ \text{excluding } \lambda_l = 1 \text{ and } c_b = c_\lambda}} \phi_{\mu,\{2\}} \\
& + \underset{\substack{\lambda/\mu \text{ a vertical pair,} \\ \lambda_l = 1,\, c_b = c_\lambda,\, \lambda_{l-1} = 2, \\ \text{one choice of sign}}}{\phi^{\pm}_{\mu,\{2\}}} \\
& + \sum_{\substack{\lambda/\mu \text{ a horizontal pair,} \\ \text{excluding } \lambda_l = 2 \text{ and } \lambda/\mu \text{ the bottom row}}} \phi_{\mu,\{2\}} \\
& + \underset{\substack{\lambda/\mu \text{ a horizontal pair,} \\ \lambda_l = 2,\, \mu \text{ the bottom row,} \\ \text{one choice of sign}}}{\phi^{\pm}_{\mu,\{2\}}}.
\end{aligned}$$

The condition "one choice of sign" occurring in (ii) is imposed because of the exceptional case explained in 2E.

The main result of this section is the following theorem.

THEOREM 4.3. *Suppose $n \geqslant 5$, $\lambda \neq \{n\}$, and $\lambda \neq \{l, l-1, ..., 1\}$, and let s, t be as in* (4.1). *Then*

(i) $t \geqslant 1$.

(ii) *If λ is even and $\lambda \neq \{l+r, l-1+r, ..., 1+r\}$, then $t \geqslant 2$.*

(iii) *If $\lambda \neq \{l+r, l-1+r, ..., 1+r\}$, then $s \geqslant 1$.*

Proof. The analysis uses the branching rules to determine the various norms needed to apply Proposition 4.1. The branching theorem tells precisely the number of constituents of $\phi_\lambda \mid \hat{S}_{n-1,1}$. Indeed, suppose λ has exactly x corners in its shifted diagram. For each corner c, let $\mu = \lambda/c$ be the partition whose shifted diagram is obtained by removing c. Then $\lambda/c \in DP_{n-1}$ and $\phi_{\lambda/c,\{1\}}$ is an irreducible constituent. If λ is odd, λ/c is even, unless $\lambda_l = 1$ and c is the bottom corner, in which case the exception applies. The branching rule shows that $\phi_{\lambda/c,\{1\}}$ is a constituent of $\phi_\lambda \mid \hat{S}_{n-1,1}$. In the exceptional case, only one of the irreducibles $\phi^{\pm}_{\lambda/c,\{1\}}$ occurs. In particular, there are always x constituents when λ is odd. As the constituents of $\phi_\lambda \mid \hat{S}_{n-1,1}$ are distinct (see (2.8)), its norm is precisely x

when λ is odd, and so here $s = x - 1$. If λ is even, λ/c is odd, except if $\lambda_l = 1$ and c is the bottom corner. Each odd λ/c contributes two distinct irreducibles labelled $\phi^{\pm}_{\lambda/c, \{1\}}$. The case $\lambda_l = 1$ contributes just one irreducible $\phi_{\lambda/c}$ as λ/c is even. In all, there are $2x$ irreducibles if $\lambda_l \neq 1$ or $2x - 1$ if $\lambda_l = 1$. This shows (again using the fact that the constituents of $\phi_\lambda | \hat{S}_{n-1,1}$ are distinct) that $s = 2x - 1$ if $\lambda_l \neq 1$ and $2x - 2$ if $\lambda_l = 1$. In particular, when $\lambda \neq \{l + r, l - 1 + r, ..., 1 + r\}$, we have $x \geqslant 2$, and so $s \geqslant 1$, proving (iii).

We now prove (i) and (ii). Suppose first that λ is even. Recall the norm of $\phi_\lambda | \hat{S}_{n-1,1}$ is $2x$ if $\lambda_l \neq 1$ and $2x - 1$ if $\lambda_l = 1$. Moreover, the number of pairs of corners is $x(x-1)/2$. If $x \geqslant 3$ there are at least as many pairs of corners as corners. According to Lemma 4.2(i), for each pair there is at least one irreducible constituent in $\phi_\lambda | \hat{S}_{n-2,2}$ of the form $\phi_{\mu, \{2\}}$ with λ/μ a disjoint pair and with multiplicity 2. Hence the norm of $\phi_\lambda | \hat{S}_{n-2,2}$ is at least $4x$. By Proposition 4.1, $t \geqslant (4x - 2x) \geqslant 6$, as desired. Suppose $x = 2$. There is one disjoint pair contributing 8 to the norm if $\lambda_l \neq 1$ and 4 if $\lambda_l = 1$. In the first case, by Proposition 4.1, $t \geqslant 8 - 4$. In the second case, as $n \geqslant 5$, there is also either a vertical or a horizontal pair contributing at least 1 to the sum in Lemma 4.2(i). Now $t \geqslant 5 - 3 = 2$. Suppose $x = 1$. Our hypotheses ensure that $\lambda_l \geqslant 2$ and $l \geqslant 2$, and so there is a vertical pair and a horizontal pair. The vertical pair contributes $\phi^{\pm}_{\mu, \{2\}}$ and the horizontal pair contributes 1 or 2 constituents depending on whether $\lambda_l = 2$. At any rate, the norm is 3 or 4, and so again $t \neq 0$ by Proposition 4.1.

Suppose now that λ is odd. In this case, the norm of $\phi_\lambda | \hat{S}_{n-1,1}$ is x. Suppose $x \geqslant 3$. There are at least x pairs of disjoint corners, each contributing 2 or 4 to the norm in Lemma 4.2(ii). This means the norm of $\phi_\lambda | \hat{S}_{n-2,2}$ is at least $2x$, and so $t \geqslant 2x - x > 0$ by Proposition 4.1. Suppose $x = 2$. There is a contribution of at least 2 to the norm from the disjoint pair. As $n \geqslant 5$, there is either a horizontal or a vertical pair as well, forcing $t \neq 0$, again. Finally, suppose $x = 1$. As $l \geqslant 2$ and $\lambda_l \geqslant 2$, there are horizontal and vertical pairs giving a norm of at least 2. The theorem has been proven. ∎

5. The Main Reduction

In this section we combine the information obtained so far with certain results in [15] to show that most faithful characters of $\hat{S}_n$ are reducible when they are restricted to proper subgroups which are not highly transitive.

THEOREM 5.1. *Suppose ϕ_λ is a faithful character of $\hat{S}_n$ and $\hat{H}$ is a proper subgroup for which $\phi_\lambda | \hat{H}$ is irreducible. Then one of the following holds.*

(i) *H is 2-homogeneous.*

(ii) $\lambda = \{n\}$.

(iii) $\lambda = \{l, l-1, ..., 1\}$.

(iv) $H \leqslant S_{n-1,1}$, *H is 2-homogeneous on* $\{1, ..., n-1\}$, $\lambda = \{l+r, l-1+r, ..., 1+r\}$, *with* $l \geqslant 2$, $r \geqslant 1$, *and* λ *odd.*

Proof. The case $n \leqslant 4$ can be handled easily by inspection, so we take $n \geqslant 5$. Assume that (i), (ii), and (iii) fail. Let s and t be as in Proposition 4.1, and observe that $t \neq 0$ by Theorem 4.3(i). Now as $\phi_\lambda | \hat{H}$ is irreducible, $\langle \psi_n + s\psi_{n-1,1} + t\psi_{n-2,2} | H, 1 \rangle_H = \langle \phi_\lambda \bar{\phi}_\lambda | H, 1 \rangle_H = 1$. So because $t \neq 0$, we have $\langle \psi_{n-2,2} | H, 1 \rangle_H = 0$. Hence as (i) fails, [15, Proposition 8.3] ensures that H fixes one point and is 2-homogeneous on the remaining $n-1$ points. If λ has more than one corner, $s \neq 0$ by Theorem 4.3(iii) and so $\phi_\lambda \bar{\phi}_\lambda$ contains $\psi_{n-1,1}$ as a constituent. Moreover, by [15, Proposition 8.1], $\langle \psi_{n-1,1} | H, 1 \rangle_H \neq 0$ as H is intransitive. This forces $\langle \phi_\lambda \bar{\phi}_\lambda | H, 1 \rangle_H \geqslant 2$, a contradiction. Therefore λ has a unique corner, c, and since (iii) fails, we have $\lambda_l \geqslant 2$. Suppose for the moment that λ is even. Then by the branching rule in (2.8), $\phi_\lambda | \hat{S}_{n-1} = \phi^+_{\lambda, c} + \phi^-_{\lambda, c}$, which forces $\phi_\lambda | \hat{H}$ to be reducible, a contradiction. Therefore λ is odd, and hence (iv) holds. Note that $r \geqslant 1$ as (iii) fails. ∎

The rest of this paper is devoted to studying the four cases described by Theorem 5.1. Clearly 2-homogeneous groups are primitive, and the primitive subgroups are analysed in Section 6. In Sections 7 and 8 we determine precisely when $\phi_{\{n\}}$ and $\phi_{\{l, l-1, ..., 1\}}$ remain irreducible upon restriction to a maximal subgroup of $\hat{S}_n$. If H is assumed to be a maximal subgroup, according to the hypothesis in Theorem 1.1, part (iv) of Theorem 5.1 leads to the subgroups listed in part (2) of Theorem 1.1 for which λ is odd and $r \geqslant 1$. Note that the branching rule (2.8) shows that $\phi_{\{l+r, l-1+r, ..., 1+r\}} | \hat{S}_{n-1}$ is indeed irreducible when λ is odd, $l \geqslant 2$, and $r \geqslant 1$.

6. Primitive Subgroups

The main results in this section are Theorems 6.3 and 6.4, which classify the situations in which a faithful irreducible character of $\hat{S}_n$ or $\hat{A}_n$ remains irreducible upon restriction to a primitive subgroup. Obviously a 2-homogeneous group is primitive, and hence this section serves to handle part (i) of Theorem 5.1.

As we see in the proof of Theorems 6.2 and 6.3, below, it is important to our analysis to classify the primitive subgroups H of S_n with order at least 2^{n-4}. The determination of the large primitive subgroups has a rich history, and a good source of reference is the work of Cameron [4, Sect. 6]. Using the O'Nan–Scott Theorem (see [2, 4, 12, 20]), one quickly reduces to the case where H is *almost simple*, that is, $F^*(H)$ is non-abelian simple. Then by using the classification of the finite simple groups, together with the determination of their large subgroups, we can obtain all possibilities for H.

Throughout this section, for a group X we write $P(X)$ for the smallest degree of a faithful permutation representation of X. In other words,

$$P(X) = \min\{k \mid X \text{ embeds in } S_k\}.$$

PROPOSITION 6.1. *Assume that S is a non-abelian proper simple subgroup of A_n and that $|\mathrm{Aut}(S)| \geqslant 2^{n-4}$. Then one of the following holds.*

(i) *$S \cong A_m$ for some m, and if $m \geqslant 9$, then each orbit of S has size* 1 *or m. In particular, if $m \geqslant 9$, S is intransitive. If $m = 5$, 6, 7, or 8, then $n \leqslant 10$,* 14, 16, *or* 19, *respectively.*

(ii) *$S \cong L_2(q)$ with $q = 7$, 8, 11, 13, or 16, and $n \leqslant 12$, 14, 14, 15, or* 17, *respectively.*

(iii) *$S \cong L_3(3)$ or $L_3(4)$, and $n \leqslant 17$ or* 21, *respectively.*

(iv) *$S \cong M_t$ with $t = 11$, 12, 22, 23, or 24, and $n \leqslant 16$, 21, 23, 27, or* 31, *respectively.*

Proof. Obviously $n \geqslant P(S)$, and hence

$$|\mathrm{Aut}(S)| \geqslant 2^{n-4} \geqslant 2^{P(S)-4}. \tag{6.1}$$

Using the classification of simple groups, S is either alternating, a group of Lie type, or a sporadic simple group. Assume $S \leqslant A_n$ and let $I = \{1, ..., n\}$ be the set upon which A_n acts naturally.

First suppose that $S \cong A_m$ for some $m \geqslant 9$, so that $|\mathrm{Aut}(S)| = |S_m| = m!$. Assume for a contradiction that S has an orbit on I of size different from 1 or m. Let S_1 be the stabilizer in S of a point in this orbit, so that $|S : S_1|$ is the size of this orbit. Now it is well known that the largest subgroup of A_m other than A_{m-1} is in fact S_{m-2}, the stabilizer of a 2-set (this fact follows easily from an old theorem of Bochert [3], for example). Therefore $|S : S_1| \geqslant \binom{m}{2}$. But then

$$2^{\binom{m}{2}-4} \leqslant 2^{n-4} \leqslant |\mathrm{Aut}(S)| = m!,$$

which is absurd for $m \geqslant 9$. This contradiction shows that if $m \geqslant 9$, each orbit has size 1 or m. A quick check of orders shows the remaining bounds for n hold when $5 \leqslant m \leqslant 8$.

When S is one of the classical groups $L_m(q)$, $U_m(q)$, $PSp_m(q)$, $P\Omega_n^{\varepsilon}(q)$, we appeal to Cooperstein [6] for the values of $P(S)$. And when S is one of the exceptional groups $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $Sz(q)$, ${}^2G_2(q)$, ${}^2F_4(q)'$, ${}^2E_6(q)$, or ${}^3D_4(q)$, we can quote [11]. In all these cases, (6.1) holds only in the situations described in (ii) and (iii) in the proposition.

Consider finally the case where S is a sporadic group. Let d be the smallest degree of a non-trivial complex character of S, and observe that $d \leqslant P(S)-1$ (since a permutation character of degree m has a non-trivial constituent of degree at most $m-1$). Thus by (6.1), $|\mathrm{Aut}(S)| \geqslant 2^{d-3}$. Now the character tables of all the sporadic groups are known (see [5]), and it is easy to check that this inequality holds only if S is a Mathieu group or $S \cong J_2$, HS, McL, Co_2 or Co_3. The maximal subgroups of each of these groups have been classified (see [5] again), and hence $P(S)$ is known. Thus we can check that (6.1) fails, except when S is a Mathieu group. This completes the proof. ∎

PROPOSITION 6.2. *Assume that H is a subgroup of S_n not containing A_n and that $n \geqslant 5$. Assume further that H acts primitively and that $|H| \geqslant 2^{n-4}$. Then one of the following holds.*

(1) *$n = p^k$ for some prime p and either*

(a) *$n = 5$ or 7 and $\mathbf{Z}_n \leqslant H \leqslant \mathbf{Z}_n : \mathbf{Z}_{n-1}$;*

(b) *$n = 2^k$ for $3 \leqslant k \leqslant 5$ and $\mathbf{Z}_2^k \trianglelefteq H \leqslant AGL_d(2)$; or*

(c) *$n = 9$ and $\mathbf{Z}_3^2 \trianglelefteq H \leqslant AGL_2(3)$.*

(2) *H is almost simple and by putting $S = \mathrm{soc}(H)$, we get either*

(d) *$S \cong A_5$ and $n \in \{6, 10\}$;*

(e) *$S \cong A_6$ and $n = 10$;*

(f) *$S \cong A_7$ and $n = 15$;*

(g) *$S \cong A_8$ and $n = 15$;*

(h) *$S \cong L_2(7)$ and $7 \leqslant n \leqslant 8$;*

(i) *$S \cong L_2(q)$, $q = 8, 13, 16$, and $n = q+1$;*

(j) *$S \cong L_2(11)$ and $11 \leqslant n \leqslant 12$;*

(k) *$S \cong L_3(3)$ and $n = 13$;*

(l) *$S \cong L_3(4)$ and $n = 21$;*

(m) *$S \cong M_{11}$ and $11 \leqslant n \leqslant 12$; or*

(n) *$S \cong M_t$ and $n = t$ for $t = 12, 22, 23, 24$.*

Proof. We begin by appealing to the O'Nan–Scott Theorem. For convenience, we use the description of this theorem as it appears in [12]. If H is of type I (in the sense of [12]), then $n = p^k$ for some prime p and some k, and H contains a regular normal elementary abelian subgroup of order p^k. In addition, $\mathbf{Z}_p^k \trianglelefteq H \leqslant AGL_k(p) \cong p^k \cdot GL_k(p)$, and so $|H| \leqslant p^{k+k^2}$. Therefore

$$2^{p^k-4} \leqslant p^{k+k^2},$$

which implies

$$(k, p) \in \{(1, 5), (1, 7), (2, 3), (3, 2), (4, 2), (5, 2)\},$$

and these possibilities are listed in part (1) of the theorem.

Next take H to be of type III(b). Then $n = a^b$ and $|H| \leqslant |S_a \wr S_b| = (a!)^b\, b!$. Consequently

$$2^{a^b - 4} \leqslant (a!)^b\, b!,$$

which is impossible as $a \geqslant 5$. (To see this, observe that $(a!)^b\, b! \leqslant (a^a)^b\, b^b \leqslant n^a b^b \leqslant n^{\sqrt{n}} \log_5(n)^{\log_5(n)}$, and for $n \geqslant 49$, we have $2^{n-4} > n^{\sqrt{n}} \log_5(n)^{\log_5(n)}$. Therefore $n \leqslant 36$, and a direct check eliminates $n = 36$ and 25.)

Now take H to be of type III(a). Here $n = |T|^{k-1}$ with T a non-abelian simple group, and H embeds in a group with structure $T^k \cdot (S_k \times \mathrm{Out}(T))$ (that is, the direct product of k copies of T extended by the direct product $S_k \times \mathrm{Out}(T)$). Note that $|T| \geqslant 60$ and $|\mathrm{Out}(T)| < |T|$ (the second fact follows from the classification), and hence

$$2^{n-4} = 2^{|T|^{k-1} - 4} > |T|^{k+1}\, k! \geqslant |T|^k\, |\mathrm{Out}(T)|\, k! \geqslant |H|,$$

a contradiction.

If H is of type III(c), then H is actually contained in a group of type III(a) or III(b), and so no examples arise here, either.

We are now left to consider the case where H is of type II. Thus H is almost simple, and we write S for the socle $\mathrm{soc}(H)$, so that S is a non-abelian simple group. Clearly S satisfies the hypotheses in Proposition 6.1, and hence is one of the groups appearing in that proposition. Moreover, as H is primitive, S is transitive. Thus if S is an alternating group A_m, then $m < 9$ by Proposition 6.1(i). The only task remaining is to find those allowable values of n corresponding to a given simple group. Since the maximal subgroups of all groups appearing in Proposition 6.1 are known (all are listed in [5], for example), this task is an elementary exercise. For example, if $S \cong A_6$, then $|H| \leqslant |\mathrm{Aut}(S)| = 1440 < 2^{11}$, and so $n \leqslant 14$. Now the only non-trivial transitive actions of A_6 of degree at most 14 are those of degrees 6 and 10. Since H does not contain A_n (by assumption), we must have $n = 10$, as given in part (e) above. The remaining cases are just as easy, and we leave these to the reader. ∎

We are now in a position to prove the main results of this section. These two theorems are proved simultaneously.

THEOREM 6.3. *Assume that H is a primitive subgroup of S_n not containing A_n, with $n \geqslant 5$. Then the character ϕ_λ of $\hat{S}_n$ remains irreducible upon restriction to $\hat{H}$ if and only if n, H, and λ appear in Table* I.

TABLE I

n	H	λ
5	$\mathbf{Z}_5 : \mathbf{Z}_4$	$\{5\}, \{3,2\}$
6	S_5	$\{6\}, \{3, 2, 1\}$
6	A_5	$\{6\}$
8	$AGL_3(2)$	$\{8\}$
10	S_6, M_{10}, $\mathrm{Aut}(A_6)$	$\{10\}$
12	M_{12}	$\{12\}$

THEOREM 6.4. *Assume that H is a primitive proper subgroup of A_n, with $n \geqslant 5$. Then the character η_λ of $\hat{A}_n$ remains irreducible upon restriction to $\hat{H}$ if and only if n, H, and λ appear in Table* II.

Remarks. In the last row of Table I, both the S_6 and the M_{10} are contained in the $\mathrm{Aut}(A_6)$. In row six of Table II, there are two classes of subgroups $L_2(8)$, with representatives H_1, H_2, say, and two characters $\eta^1_{\{9\}}$, $\eta^2_{\{9\}}$. The indices may be chosen so that $\eta^i_{\{9\}} \mid H_i$ is irreducible while $\eta^{3-i}_{\{9\}} \mid H_i$ is reducible, for $i = 1, 2$.

Proofs of Theorems 6.3 *and* 6.4. Throughout, write $I = \{1, ..., n\}$ for the set upon which S_n acts naturally. Let G be either A_n or S_n, let $\lambda \in \mathrm{DP}_n$, and let χ_λ denote ϕ_λ or η_λ according to whether G is S_n or A_n. Note that Theorem 3.1 and Corollary 3.2 show that any faithful irreducible character of $\hat{G}$ has degree at least $2^{(n-3)/2}$, and so $2^{n-3} \leqslant |\hat{H}|$. Consequently if $\chi_\lambda \mid \hat{H}$ is irreducible, then $2^{n-4} \leqslant |H|$, and hence H and n appear in Proposition 6.2. Thus to prove the theorems, it suffices to go through the list of pairs H, n in Proposition 6.2 and determine precisely which characters ϕ_λ or η_λ remain irreducible when restricted to $\hat{H}$. The case $n = 5$ is an easy exercise, so we take $n \geqslant 6$. As usual, we write S for the socle of H. In several

TABLE II

n	H	λ
5	D_{10}	$\{5\}$
6	A_5	$\{6\}$
7	$L_2(7)$ (two classes)	$\{7\}$
8	$AGL_3(2)$ (two classes)	$\{8\}, \{7, 1\}$
9	$3^2 : Q_8 \leqslant H \leqslant 3^2 : SL_2(3)$	$\{9\}$
9	$L_2(8) \leqslant H \leqslant \mathrm{Aut}(L_2(8))$	$\{9\}$
10	M_{10}	$\{10\}$
11	M_{11} (two classes)	$\{11\}$
12	M_{12} (two classes)	$\{12\}, \{11, 1\}$

instances, we make use of the fact that an involution in A_n lifts to an involution in $\hat{A}_n$ if and only if it is a product of $4s$ disjoint transpositions for some integer s.

Case $n=6$. Here H is either A_5 or S_5 by Proposition 6.2, and the action of H on I is isomorphic to its action on its Sylow 5-subgroups. Take $H=S_5$. Then $G=S_6$, since $S_5 \nleq A_6$. Because the preimage of $(12)(34)$ in $\hat{S}_6$ has order 4, $\hat{H}$ is a non-splitting double cover of S_5. Observe that a transposition in S_5 fixes no Sylow 5-subgroup, and so its image in S_6 is a product of three disjoint transpositions. Such an involution lifts to an involution in $\hat{S}_6$, and so $\hat{H}$ is in fact the non-splitting double cover of S_5 other than $\hat{S}_5$, as discussed in Section 1. In view of the correspondence between the faithful characters of $\hat{S}_5$ and those of the other non-splitting double cover of S_5 (also mentioned in Section 1), we can use the same notation ϕ_μ $(\mu \in \mathrm{DP}_5)$ for faithful irreducible characters of $\hat{H}$. Since $\phi_{\{5,1\}}(1)=16$, $\phi_{\{4,2\}}(1)=20$, $\phi_{\{6\}}(1)=\phi_{\{3,2,1\}}(1)=4$, it is clear from degree considerations that row two of Table I holds, as the smallest faithful degree for $\hat{H}$ is 4. Furthermore, it follows from the character values given in [22, (3.3)] that $\phi_{\{6\}} \mid \hat{H}=\phi_{\{3,2\}}$ and $\phi_{\{3,2,1\}} \mid \hat{H}=\phi_{\{5\}}$. Of these, $\phi_{\{3,2\}} \mid \hat{A}_5$ is irreducible but $\phi_{\{5\}} \mid \hat{A}_5$ is reducible. This gives row three of Table I. The same arguments can be applied to obtain row two of Table II. (Recall that this subgroup S_5 is acting transitively on six letters, and hence the usual branching rule does not apply.) This completes the analysis for $n=6$.

Case $n=7$. As the irreducible character degrees of subgroups of $\mathbf{Z}_7:\mathbf{Z}_6$ divide 6 by Ito's Theorem, while none of $\hat{G}$ divide 6, H cannot be contained in $\mathbf{Z}_7:\mathbf{Z}_6$. Therefore $S \cong L_2(7)$. Since $\mathrm{Aut}(L_2(7)) \nleq S_7$, it follows that $H \cong L_2(7)$, and hence $\hat{H} \cong SL_2(7)$ (since involutions in A_7 lift to elements of order 4 in $\hat{A}_7$). The degrees of faithful irreducible characters of $SL_2(7)$ are 4, 6, and 8 (see [18]), and so the only possibility is $\lambda=\{7\}$, which is shown in Table II, row three. Here $\eta_{\{7\}}$ has degree 4, and so $\eta_{\{7\}} \mid \hat{H}$ is indeed irreducible.

Case $n=8$. Suppose first that $\mathbf{Z}_2^3 \trianglelefteq H \leqslant AGL_3(2) \cong \mathbf{Z}_2^3:GL_3(2)$, a semidirect product of $\mathbf{Z}_2^3$ by $L=GL_3(2)$. Thus $H=\mathbf{Z}_2^3:K$, where K is a subgroup of L. Here I may be identified with the elements of the normal $\mathbf{Z}_2^3$. The normal $\mathbf{Z}_2^3$ acts on itself by multiplication, and L acts by conjugation. Thus K is the point stabilizer in H, and since H is primitive, K is maximal in H. Therefore, K acts irreducibly on the $\mathbf{Z}_2^3$, regarded as a 3-dimensional vector space over $\mathbf{F}_2$. This means H is not a $\{2,3\}$-group, and so $2^3:7 \leqslant H$. Moreover, since all involutions in $O_2(H)$ act fixed point freely, their preimages in $\hat{S}_8$ are also involutions. Therefore $\widehat{O_2(H)}$ is abelian, and so Ito's Theorem implies that $\chi_\lambda(1) \mid |H:O_2(H)|$. Thus the only possibility is $\chi_\lambda=\phi_{\{8\}}$, $\eta_{\{8\}}$, or $\eta_{\{7,1\}}$, and hence $8 \mid |H:O_2(H)|$. It now follows that

$H = AGL_3(2)$. The group $O_2(\hat{H})$ is elementary abelian of order 16, and we claim that it is indecomposable as an $\mathbf{F}_2\hat{H}$-module. For suppose otherwise. Then $\hat{H}$ contains a normal elementary abelian subgroup E of order 8, and it follows from Clifford's Theorem that any irreducible character of $\hat{H}$ whose kernel does not contain E must have degree divisible by 7. Consequently $\phi_{\{8\}} | \hat{H}$ has irreducible constituents of degrees 1 and 7. On the other hand, L contains an involution centralizing a 2-space in $O_2(H)$, and so has cycle type $(2, 2)$ in S_8. The preimage of this involution has order 4 in $\hat{S}_8$, whence $\hat{L} \cong SL_2(7)$. Therefore every irreducible constituent of $\phi_{\{8\}} | \hat{L}$ has even degree, which contradicts the fact that $\phi_{\{8\}} | \hat{H}$ has irreducible constituents of degrees 1 and 7. This contradiction shows that $\widehat{O_2(H)}$ is indeed indecomposable. It follows that $\hat{H}$ has just two orbits on the 15 non-trivial linear characters of $\widehat{O_2(H)}$, namely, the 7 which contain $Z(\hat{S}_8) = Z(\hat{H})$ in their kernel and the 8 which do not. Let us write ρ for the sum of the characters in this orbit of size 8. Now consider an irreducible character χ of $\hat{H}$ for which $Z(\hat{H})$ acts non-trivially. Since $\widehat{O_2(H)}$ is indecomposable, $Z(\hat{H})$ is the unique minimal normal subgroup of $\hat{H}$, and so χ is faithful. Therefore Clifford's Theorem shows that $\chi | \widehat{O_2(H)} = e\rho$ for some integer e. In particular $8 | \chi(1)$. We now see that $\phi_{\{8\}}$ and $\eta_{\{8\}}$ remain irreducible upon restriction to $\hat{H}$, as displayed in row four of Table I and row four of Table II. Further, when restricted to $\hat{H}$, the character $\eta_{\{7,1\}}$ has either three irreducible constituents of degree 8, one of degree 8 and one of degree 16, or is irreducible. The second possibility is impossible by Ito's Theorem. The first possibility cannot occur either, for any faithful irreducible character of $\hat{H}$ of degree 8 must assume the value 1 on an element of order 7, and $\eta_{\{7,1\}}$ does not assume the value 3 here. Therefore $\eta_{\{7,1\}} | \hat{H}$ is irreducible, as shown in row four of Table II.

Next assume that $S \cong L_2(7)$. Here the action of H on I is isomorphic to its action on its Sylow 7-groups, and so elements of order 2 and 3 in S have cycle types $(2, 2, 2, 2)$ and $(3, 3)$ in S_8. In particular, the preimage of each involution in S is also an involution in $\hat{S}$, and so $\hat{S} \cong \mathbf{Z}_2 \times L_2(7)$. Here the only possibility is $\lambda = \{8\}$, but in view of the cycle types just mentioned, it is clear that $\phi_{\{8\}} | \hat{S}'$ has constituents of degrees 1 and 7. It now follows that $\phi_{\{8\}} | \hat{H}$ and $\eta_{\{8\}} | \hat{H}$ are reducible, and so no example occurs.

Case $n = 9$. First assume that $H \leqslant AGL_2(3) \cong 3^2 : GL_2(3)$. Since H is primitive, $O_3(AGL_2(3)) = O_3(H)$ and H acts irreducibly on $O_3(H)$ (this is analogous to the analysis of $n = 8$ and $H = AGL_3(2)$). Obviously $\widehat{O_3(H)}$ is abelian, and so Ito's Theorem implies that $\phi_\lambda(1) \,|\, |H : O_3(H)|$. Thus $\phi_\lambda(1) | 48$, and so if $G = S_9$, then $\lambda = \{9\}$ with $\phi_{\{9\}}(1) = 16$. If $G = A_9$, then

$H \leqslant 3^2 : SL_2(3)$, which means $\phi_\lambda(1) \mid 24$; once again $\lambda = \{9\}$. We observe that $\phi_{\{9\}} \mid \hat{A}_9 = \eta^1_{\{9\}} + \eta^2_{\{9\}}$, and these two irreducible characters of degree 8 differ only on the classes corresponding to 9-cycles. When $G = A_9$, we must have $8 \mid |H : O_3(H)|$, and so $3^2 : Q_8 \leqslant H \leqslant 3^2 : SL_2(3)$. Since the Q_8 acts transitively on the eight non-trivial linear characters of $O_3(\hat{H})$, it follows that any irreducible representation of $\hat{H}$ for which $O_3(\hat{H})$ acts non-trivially must have degree at least 8. Therefore we obtain examples as given in row six of Table II. Now take $G = S_9$. If $\phi_{\{9\}} \mid \hat{H}$ is irreducible, then it must be the case that $\phi_{\{9\}} \mid (\hat{H} \cap \hat{A}_9)$ is a sum of two distinct irreducible characters. On the other hand, H contains no elements of order 9, and since $\eta^1_{\{9\}}$ and $\eta^2_{\{9\}}$ differ only on elements of order 9, we are forced to conclude that $\eta^1_{\{9\}} \mid (\hat{H} \cap \hat{A}_9) = \eta^2_{\{9\}} \mid (\hat{H} \cap \hat{A}_9)$. Consequently it cannot be the case that $\phi_{\{9\}} \mid \hat{H}$ is irreducible.

Now we take the case where $S \cong L_2(8)$. By considering degrees, we find that the only possibility is $\lambda = \{9\}$ and $G = A_9$. The action of H on I is isomorphic to the action on its Sylow 2-subgroups, and so elements of order 2 and 3 in S have cycle types $(2, 2, 2, 2)$ and $(3, 3, 3)$, respectively. Further, since S has trivial multiplier, $\hat{S} \cong \mathbf{Z}_2 \times L_2(8)$. Using this information, one checks that $\eta^i_{\{9\}} \mid \hat{S}'$ is either irreducible or reducible according to which class in A_9 the elements of order 9 in $\hat{S}'$ fuse. There are just two classes in A_9 of elements of order 9 which consist of 9-cycles; these two classes are conjugate in S_9. There are also precisely two conjugacy classes of subgroups isomorphic to $L_2(8)$ in $\hat{A}_9$, with representatives H_1, H_2, say. The elements of order 9 in H_1 and H_2 are 9-cycles which lie in different A_9-classes, and $\eta^1_{\{9\}}$ and $\eta^2_{\{9\}}$ differ on these classes. The restriction of $\eta^i_{\{9\}}$ to one of the groups H_j is irreducible and to the other it has constituents of degrees 1 and 7. Thus for each $i = 1, 2$, there is a unique $\hat{A}_9$-class of subgroups $L_2(8)$ to which $\eta^i_{\{9\}}$ restricts irreducibly. Thus row six of Table II holds.

Case $n = 10$. Here S is either A_5 or A_6. It is made obvious, however, that $S \not\cong A_5$ simply by considering degrees. So assume that $S \cong A_6$, and observe that the action of S on I is isomorphic to the action on its Sylow 3-subgroups. As before, we determine the class fusion from A_6 to S_{10}, and we find that the elements of orders 2, 3, 4, and 5 in S have cycle types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(4, 4)$, and $(5, 5)$ in S_{10}. Thus $\hat{S} \cong \mathbf{Z}_2 \times A_6$, and hence the degree of any irreducible character of $\hat{H}$ is at most $|\mathrm{Aut}(S) : S| = 4$ times the degree of an irreducible character of S. Thus $\chi_\lambda(1) \leqslant 40$, and so it must be the case that $\lambda = \{10\}$, as other character degrees are at least 48. According to the fusion described above, we see that $\chi_{\{10\}} \mid \hat{S}'$ is a sum of two distinct characters of degree 8. It follows that $\chi_{\{10\}} \mid \hat{H}$ is irreducible if and only if $\hat{H}$ induces an automorphism on $\hat{S}'$ which interchanges these two characters. Recall that the three subgroups of index 2 in $\mathrm{Aut}(S)$ are S_6,

$PGL_2(9)$, and M_{10}, and $PGL_2(9)$ is in fact the stabilizer of each of these characters in $\mathrm{Aut}(S)$. Consequently, $S \leqslant H \nleqslant PGL_2(9)$, and so H is one of S_6, M_{10}, and $\mathrm{Aut}(S)$. Since elements of $S_6 \backslash S$ induce odd permutations on the Sylow 3-subgroups of S, we now see that row five of Table I and row seven of Table II give the correct examples for $n = 10$. Note that M_{10} lies in A_{10} whereas S_6 and $PGL_2(9)$ do not.

Case $n = 11$. Here S is either $L_2(11)$ or M_{11}. However, it is easy to discard $L_2(11)$, for the largest degree of an irreducible character of a covering group of $\mathrm{Aut}(L_2(11))$ is 12. Thus we must have $S \cong M_{11}$, and by considering degrees, we see that the only possibility is for $G = A_{11}$ and $\lambda = \{11\}$. Now one can read from the permutation character of M_{11} that each involution fixes just three points in I. Thus $\eta_{\{11\}}$ takes the value 0 on involutions in M_{11}, and hence it follows that $\eta_{\{11\}} | \hat{S}'$ is irreducible. As there are just two classes of M_{11} in A_{11}, row eight of Table II has now been verified.

Case $n = 12$. Here $S \cong L_2(11)$, M_{11}, or M_{12}. The usual degree considerations eliminate $L_2(11)$ and M_{11}. By using the permutation characters of M_{12}, we see that involutions with centralizer $\mathbf{Z}_2 \times S_5$ in M_{12} are fixed point free, and hence the preimages in $\hat{S}_{12}$ of such involutions have order 4. Consequently $\hat{S} \cong 2 \cdot M_{12}$, the double cover of M_{12}. By comparing the degrees of the irreducible characters of $\hat{A}_{12}$ and $\hat{S}_{12}$ and those of $2 \cdot M_{12}$, we see that the only possibilities are $\chi_\lambda = \phi_{\{12\}}$, $\eta_{\{12\}}$, or $\eta_{\{11,1\}}$. By using the permutation characters, one can easily determine all class fusion from M_{12} to S_{12}, and thence verify that these characters indeed remain irreducible upon restriction to $2 \cdot M_{12}$. We thus obtain row six of Table I and row nine of Table II.

Case $n = 13$. Here $S \cong L_3(3)$, and the action of S on I is isomorphic to the action of S on the 1-spaces in the 3-dimensional vector space over $GF(3)$. Since this action does not extend to $\mathrm{Aut}(L_3(3))$, it must be the case that $H = S$. But then $\hat{H} \cong \mathbf{Z}_2 \times L_3(3)$ has no faithful irreducible character with the same degree as one of $\hat{A}_{13}$ or $\hat{S}_{13}$.

Case $n = 14$. Here $S \cong L_2(13)$. But then the largest degree of an irreducible character of a covering group of $\mathrm{Aut}(S)$ is only 14, and so no example can arise.

Case $n = 15$. Here $S \cong A_7$ or A_8. Degree considerations eliminate A_7. So we analyze the case $S \cong A_8$. The action of S on I is isomorphic to its action on non-zero vectors in the 4-dimensional vector space over $GF(2)$ (recall that $A_8 \cong L_4(2)$). This action does not extend to S_8, and hence $H = S$. An involution with two Jordan blocks fixes precisely 3 non-zero vectors, and hence its preimage in $\hat{S}_{15}$ has order 4. Therefore $\hat{S} \cong \hat{A}_8$.

Moreover, it is clear (by degrees) that the only possibility is $G = A_{15}$, $\lambda = \{15\}$, and $\eta_{\{15\}} \mid \hat{A}_8 = \eta_{\{5,2,1\}}$. Now let $x \in \hat{S}$ be a 3-cycle in A_8. On the one hand we see that $\eta_{\{5,2,1\}}(x) = 4$. On the other hand, x acts with no fixed point on the 15 non-zero vectors, and so has five orbits. Therefore, using Schur's formula for the character values of the basic spin representation [18], $\eta_{\{15\}}(x) = -2$. We have proved that no example occurs in this case.

Case $n = 16$. Here $H \leqslant AGL_4(2) \cong 2^4 : L_4(2)$. As in the case where $n = 8$, we see that $\chi_\lambda(1)$ must divide $|L_4(2)|$. However, using Wagner's Theorem 2.5, $2^7 \mid \chi_\lambda(1)$, and so no example arises.

Case $n = 17$, 21, 22, 23, *or* 24. Here $S \cong L_2(16)$, $L_3(4)$, M_{22}, M_{23}, or M_{24}, respectively. In all cases, it is clear from the character degrees of S and any double cover of S that no example can arise. This is an easy yet tedious check. Alternatively, one may invoke Ito's Theorem and Wagner's Theorem 2.5. For example, if $S \cong L_3(4)$, then the action of S on I is isomorphic to its action on 1-spaces in the 3-dimensional vector space over $GF(4)$. Since this action does not extend to S with its graph automorphism adjoined, it follows that $H \leqslant P\Gamma L_3(4) \cong PGL_3(4).2$. Thus Ito's Theorem implies that $\chi_\lambda(1)$ must divide $2^7 \cdot 3^3 \cdot 5 \cdot 7$. On the other hand, Theorem 2.5 implies that $\chi_\lambda(1)$ is divisible by 2^8, and this supplies the desired contradiction. The same argument also disposes of $L_2(16)$, M_{22}, and M_{23}. A similar argument works for M_{24}, since M_{24} has no irreducible character with degree divisible by 1024.

Case $n = 32$. Here $H \leqslant AGL_5(2) \cong 2^5 : L_5(2)$, and as in the cases where $n = 8$, 16, we find that $\chi_\lambda(1)$ must divide $|L_5(2)|$. On the other hand, Theorem 2.5 implies that $\chi_\lambda(1)$ is divisible by 2^{15}, and so no example occurs.

This completes the proof of Theorems 6.3 and 6.4. ∎

7. The Staircase

In this section we consider the situation in which the character ϕ_λ remains irreducible when restricted to a maximal subgroup, where $\lambda = \{l, l-1, ..., 1\}$ with $n = l(l+1)/2$. This character is sometimes called the *staircase* because of the shape of its shifted diagram. The main result of this section is Theorem 7.1, which appears below.

Suppose $\hat{H}$ is a maximal subgroup of $\hat{S}_n$ for which $\phi_\lambda \mid \hat{H}$ is irreducible. If H is primitive, then either $H = A_n$ or H appears in Theorem 6.3, and hence H appears in Theorem 7.1(i) or (vi), below.

Now consider the case in which H is transitive yet imprimitive. Then $H = S_a \wr S_b$ where $n = ab$ and $a, b \geqslant 2$. In particular, $|H| = (a!)^b\, b!$. Using

Theorem 2.4, we can choose a prime p satisfying $n/2 < p \leqslant n$. Now primes occurring in the denominator of (2.1) are at most $2l-1$, and so if $l \geqslant 7$, we have $p > \frac{1}{4}l(l+1) \geqslant 2l-1$, which means $p \mid \phi_\lambda(1)$. However, p does not divide $|\hat{H}|$, a contradiction. Similarly, if $l=5$ or 6, then $13 \mid \phi_\lambda(1)$, yet 13 does not divide $|\hat{H}|$. We are left with the cases $l=3, 4$. If $l=4$, then $n=10$ and $\phi_\lambda(1)=96$. Since $96^2 > 2 \times (2!)^5\, 5!$, it follows that H cannot be $S_2 \wr S_5$, and hence $H = S_5 \wr S_2$, as given in part (iv) of Theorem 7.1. If $l=3$, then H appears in part (v).

Finally, suppose that H is intransitive. In this case $H = S_{n-a,a}$ with $a < n/2$. Here Corollary 2.2 shows that $a \leqslant 2$, and so H appears in Theorem 7.1(ii) or (iii).

To complete the proof of Theorem 7.1, we must show that λ does in fact remain irreducible upon restriction to the groups appearing in (i)–(vi). This holds in case (i) by 2B, and in case (vi) by Theorem 6.3. The branching rules in 2E and Lemma 4.2 take care of (ii) and (iii). We now consider part (iv). Using the method described in 2F, we see that $\phi_\lambda \mid \hat{S}_{5,5}$ has constituents $\phi_{\{3,2\},\{4,1\}}$ and $\phi_{\{4,1\},\{3,2\}}$. As these two characters have degree 48, $\phi_\lambda \mid \hat{S}_{5,5}$ is equal to their sum. Moreover, an element in $\hat{H} \backslash \hat{S}_{5,5}$ interchanges them, where $H = S_5 \wr S_2$, and so $\phi_\lambda \mid \hat{H}$ is irreducible. The same argument shows that $\phi_\lambda \mid \hat{H}$ is irreducible when $l=3$ and $H = S_3 \wr S_2$. Finally, if $H = S_2 \wr S_3$, then an outer automorphism of $\hat{S}_6$ takes $\hat{H}$ to an intransitive subgroup $\hat{S}_{4,2}$ and takes $\phi_{\{3,2,1\}}$ to $\phi_{\{6\}}$. We prove in Section 8 that $\phi_{\{6\}} \mid \hat{S}_{4,2}$ is irreducible, and hence so is $\phi_{\{3,2,1\}} \mid \hat{H}$. (Alternatively, one can show directly that this character remains irreducible.)

We have now proved

THEOREM 7.1. *Suppose that* $\lambda = \{l, l-1, ..., 1\}$, $n = l(l+1)/2 \geqslant 6$ *and that* $\hat{H}$ *is a maximal subgroup of* $\hat{S}_n$. *Then* $\phi_\lambda \mid \hat{H}$ *is irreducible if and only if one of the following holds.*

(i) $H = A_n$ *and* λ *is odd* (*i.e.*, $l \equiv 2, 3 \pmod 4$).

(ii) $H = S_{n-1,1}$.

(iii) $H = S_{n-2,2}$.

(iv) $n = 10$ *and* $H = S_5 \wr S_2$.

(v) $n = 6$ *and* $H = S_3 \wr S_2$ *or* $S_2 \wr S_3$.

(vi) $n = 6$ *and* $H = S_5$ (*transitive on six letters*).

8. THE BASIC SPIN CHARACTER

In this section we discuss situations in which the basic spin character $\phi_{\{n\}}$ remains irreducible when restricted to maximal subgroups. The main result here is

THEOREM 8.1. *Suppose $\hat{H}$ is a maximal subgroup of $\hat{S}_n$. Then $\phi_{\{n\}} \mid \hat{H}$ is irreducible if and only if one of the following holds.*

(i) *$H = A_n$ with n even (i.e., $\{n\}$ is an odd partition).*

(ii) *$H = S_{n-a,a}$ with $a < n/2$ and n even.*

(iii) *$H = S_a \wr S_b$ with $n = ab$, $a, b \geqslant 2$.*

(iv) *H is either $\mathbf{Z}_5 \cdot \mathbf{Z}_4$, S_5, $AGL_3(2)$, or $\mathrm{Aut}(A_6)$, as in Theorem 6.3.*

Before we begin the proof of Theorem 8.1, we discuss the characters of certain subgroups of S_n of the form XY, where X and Y act on disjoint subsets of $\{1, ..., n\}$. More specifically, we assume

(a) $X, Y \leqslant \hat{S}_{n-a,a}$ with $2 \leqslant a \leqslant n-2$;

(b) X acts on $\{1, ..., a\}$ and fixes $\{a+1, ..., n\}$ pointwise;

(c) Y acts on $\{a+1, ..., n\}$ and fixes $\{1, ..., a\}$ pointwise;

(d) $\langle z \rangle = Z(\hat{S}_n) = X \cap Y$;

(e) neither X nor Y is contained in $\hat{A}_n$.

Define $X_0 = X \cap \hat{A}_n$ and $Y_0 = Y \cap \hat{A}_n$, so that $[X_0, Y] = [X, Y_0] = 1$ and $[x, y] = z$ whenever $x \in X \backslash X_0$ and $y \in Y \backslash Y_0$. We generalize the notion of self-associated characters of $\hat{S}_n$ to arbitrary subgroups of $\hat{S}_n$, as follows. If χ is a character of K, a subgroup of $\hat{S}_n$, then its associate is $(\mathrm{sgn} \mid K)\chi$. If χ and its associate are equal, then χ is said to be self-associated. If $K \leqslant \hat{A}_n$, then of course all characters of K are self-associated. If χ is a self-associated character of X, then χ vanishes on $X \backslash X_0$ and $\chi \mid X_0$ is a sum of two distinct irreducibles. If χ is not self-associated, then $\chi \mid X_0$ is irreducible. Note also that if $y \in Y \backslash Y_0$, then $\chi^y = (\mathrm{sgn} \mid X)\chi$ is the associate of χ.

PROPOSITION 8.2. *Assume hypotheses* (a)–(e) *above. The irreducible characters of XY are of the form $\chi * \psi$, where $\chi \in \mathrm{Irr}(X)$, $\psi \in \mathrm{Irr}(Y)$, and the star product $\chi * \psi$ is given as follows.*

(i) *If χ and ψ are both self-associated, then $\chi * \psi$ is self-associated and*

$$\chi * \psi(xy) = \begin{cases} \chi(x)\,\psi(y) & \text{if } x \in X_0 \text{ and } y \in Y_0 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If neither χ nor ψ is self-associated, then $\chi * \psi$ is self-associated and*

$$\chi * \psi(xy) = \begin{cases} 2\chi(x)\psi(y) & \text{if } x \in X_0 \text{ and } y \in Y_0 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If χ is self-associated and ψ is not, then $\chi * \psi$ is not self-associated and*

$$\chi * \psi(xy) = \begin{cases} \chi(x)\psi(y) & \text{if } \ x \in X_0 \text{ and } y \in Y_0 \\ \Delta(x)\psi(y) & \text{if } \ x \in X_0 \text{ and } y \in Y \setminus Y_0 \\ 0 & \text{otherwise,} \end{cases}$$

where if $\chi | X_0 = \chi_1 + \chi_2$, $\Delta = \pm(\chi_1 - \chi_2)$. There are two choices for the sign of Δ. Changing the sign of Δ or taking the associate of ψ gives two associate characters.

(iv) *The same as* (iii) *with X and Y interchanged* (*so χ is not self-associated but ψ is*).

Furthermore, each of these characters is irreducible.

Proof. Suppose θ is an irreducible character of XY. Let $\chi_0\psi_0$ be an irreducible constituent of $\theta | X_0 Y_0$, where $\chi_0 \in \mathrm{Irr}(X_0)$ and $\psi_0 \in \mathrm{Irr}(Y_0)$. As X_0 and Y_0 commute, all members of $\mathrm{Irr}(X_0 Y_0)$ are of this form. By Frobenius Reciprocity, θ is a constituent of $\chi_0\psi_0 |_{X_0 Y_0}^{XY}$. Here we use a standard notation for the induced character of ϕ from H to G, namely, $\phi |_H^G$, where $H \leqslant G$ and ϕ is a character of H. Observe that $\chi_0 |_{X_0}^X$ is either irreducible or a sum of two distinct irreducibles, depending on whether $\chi_0 = \chi_0^x$ for some $x \in X \setminus X_0$. If $\chi_0^x \neq \chi_0$, $\chi_0 |_{X_0}^X$ is irreducible, in which case we say χ_0 *induces*. Let this irreducible be denoted by χ. It is self-associated, $\chi | X_0 = \chi_0 + \chi_0^x$, and $\chi_0^x |_{X_0}^X = \chi$. If $\chi_0^x = \chi_0$ for $x \in X \setminus X_0$, then $\chi_0 |_{X_0}^X$ is a sum of two distinct irreducibles which are associates. Denote them by $\chi^{\pm}$. They are not self-associated, $\chi^+ | X_0 = \chi_0$, and $\chi_0 |_{X_0}^X = \chi^+ + \chi$. In this case we say χ_0 *extends*. A similar notation is used for ψ_0. The four possibilities for χ_0 and ψ_0 inducing or extending give rise to the four cases in the proposition.

(i) Suppose χ_0 and ψ_0 both induce. Then $\psi = \psi_0 |_{Y_0}^Y$ is irreducible. As X_0 and Y commute, $\chi_0\psi \in \mathrm{Irr}(X_0 Y)$. If $x \in X \setminus X_0$, $\chi_0^x \neq \chi_0$ and $\psi^x = \psi$, as ψ is self-associated. This means $(\chi_0\psi)^x = \chi_0^x\psi^x = \chi_0^x\psi \neq \chi_0\psi$, and so $\chi_0\psi |_{X_0 Y}^{XY}$ is irreducible. The character values for $\chi_0\psi |_{X_0 Y}^{XY}$ are exactly the character values given in (i) for $\chi * \psi$. Note that χ and ψ are self-associated and $\chi * \psi$ is irreducible and self-associated. Indeed, $\chi_0\psi_0 |_{X_0 Y_0}^{XY} = \chi * \psi$ in this case. Any self-associated character χ or ψ is of the form $\chi_0 |_{X_0}^X$ or $\psi_0 |_{Y_0}^Y$, hence the character in (i) is irreducible.

(ii) Suppose χ_0 and ψ_0 both extend. Let $\chi^{\pm}$ be the two extensions of χ_0 and $\psi^{\pm}$ be those of ψ_0. Now $\chi_0\psi^+ \in \mathrm{Irr}(X_0 Y)$. If $x \in X \setminus X_0$, $\chi_0^x = \chi_0$ and $(\psi^+)^x = \psi$, and so $(\chi_0\psi^+)^x = \chi_0^x(\psi^+)^x = \chi_0\psi \neq \chi_0\psi^+$. As before, $\chi_0\psi^+ |_{X_0 Y}^{XY}$ is irreducible. Its restriction to $X_0 Y$ is $\chi_0\psi^+ + \chi_0\psi$, and so its character values are the same as those of $\chi^\varepsilon * \psi^\delta$ ($\varepsilon, \delta \in \{+, -\}$) in (ii). As neither χ nor ψ is self-associated, $\chi * \psi$ in (ii) is irreducible. Note here that $\chi_0\psi_0|^{XY}_{X_0Y_0} = 2(\chi * \psi)$. Any characters χ, ψ which are not self-associated are of the form $\chi^{\pm}$, $\psi^{\pm}$ as above, so the character in (ii) is irreducible.

(iii) Suppose χ_0 induces and ψ extends. Then $\chi_0\psi^+ \in \mathrm{Irr}(X_0 Y)$. As ψ^+ has the distinct associate ψ^- and $X_0 \leqslant \hat{A}_n$, $\chi_0\psi^+$ and $\chi_0\psi^-$ are distinct associates. Also for $x \in X\backslash X_0$, $\chi_0^x \neq \chi_0$ as χ_0 induces. Now $(\chi_0\psi^+)^x = (\chi_0^x)(\psi^+)^x = \chi_0^x\psi^- \neq \chi_0\psi^+$, and so $\gamma = \chi_0\psi^+|^{XY}_{X_0Y}$ is irreducible. Here $\gamma(xy) = 0$ if $x \in X\backslash X_0$ and $\gamma | X_0 Y = \chi_0\psi^+ + \chi_0^x\psi^-$. If $x_0 \in X_0$ and $y_0 \in Y_0$,

$$\begin{aligned}\gamma(x_0 y_0) &= \chi_0(x_0)\,\psi^+(y_0) + \chi_0^x(x_0)\psi^-(y_0)\\ &= (\chi_0 + \chi_0^x)(x_0)\psi(y_0)\\ &= \chi(x_0)\psi(y_0).\end{aligned}$$

If $x_0 \in X_0$ and $y \in Y\backslash Y_0$,

$$\begin{aligned}\gamma(x_0 y) &= \chi_0(x_0)\psi^+(y) + \chi_0^x(x)\psi^-(y)\\ &= \chi_0(x_0)\psi^+(y) - \chi_0^x(x_0)\psi^+(y)\\ &= \Delta(x_0)\psi^+(y).\end{aligned}$$

These are the character values of $\chi * \psi^+$ in (iii), and so $\chi * \psi^+$ is irreducible. Its associate is $\chi\psi^-$ or the character obtained by choosing the other sign for Δ. Here $\chi_0\psi_0|^{XY}_{X_0Y_0} = \chi * \psi^+ + \chi * \psi^-$. As above, the character in (iii) is irreducible.

(iv) If χ_0 extends and ψ induces, the argument and result are as in (iii).

We have shown that in all cases the constituents of $\chi_0\psi_0|^{XY}_{X_0Y_0}$ are of the form $\chi * \psi$, and so all irreducibles of XY are of this form. ▌

The following proposition describes how the characters $\chi * \psi$ restrict to the three subgroups of index 2, $X_0 Y$, XY_0, and $D = \langle X_0 Y_0, xy\rangle$, where $x \in X\backslash X_0$ and $y \in Y\backslash Y_0$.

PROPOSITION 8.3. *Retain the notation and hypotheses of Proposition 8.2. Also assume that $x \in X\backslash X_0$ and $y \in Y\backslash Y_0$. The restrictions of $\chi * \psi$ are as follows.*

(i) *If χ and ψ are both self-associated, then*

$$\begin{aligned}\chi * \psi \,|\, X_0 Y &= \chi_0\psi + \chi_0^x\psi\\ \chi * \psi \,|\, XY_0 &= \chi\psi_0 + \chi\psi_0^y\\ \chi * \psi \,|\, D &= (\chi_0\psi_0)^D_{X_0Y_0} + (\chi_0\psi_0^y)^D_{X_0Y_0}.\end{aligned}$$

Here $\chi | X_0 = \chi_0 + \chi_0^x$ and $\psi | Y_0 = \psi_0 + \psi_0^y$.

(ii) *If neither χ nor ψ is self-associated, then*

$$\chi * \psi \mid X_0 Y = \chi_0\psi + \chi_0\psi'$$
$$\chi * \psi \mid XY_0 = \chi\psi_0 + \chi'\psi_0$$
$$\chi * \psi \mid D = \Gamma^+ + \Gamma^-,$$

where χ' and ψ' are the associates of χ and ψ, and

$$\Gamma^{\pm} = \begin{cases} \chi(g)\psi(h) & \text{if } g \in X_0 \text{ and } h \in Y_0 \\ \pm i\chi(g)\psi(h) & \text{if } g \in X\backslash X_0 \text{ and } h \in Y\backslash Y_0. \end{cases}$$

(iii) *If χ is self-associated and ψ is not, then*

$$\chi * \psi \mid X_0 Y = \chi_0\psi + \chi_0^x\psi$$
$$\chi * \psi \mid XY_0 = \chi\psi_0$$
$$\chi * \psi \mid D \text{ is irreducible},$$

where $\psi \mid Y_0 = \psi_0$ and $\chi \mid X_0 = \chi_0 + \chi_0^x$.

Proof. All assertions follow directly from the results and proofs of the previous proposition, except for the information concerning $\chi * \psi \mid D$ in case (ii). We prove this here. Let ρ and σ be the representations affording χ and ψ, respectively, and define a map $\tau_{\pm}$ as

$$\tau_{\pm}(xy) = \begin{cases} \rho(x) \otimes \sigma(y) & \text{if } x \in X_0 \text{ and } y \in Y_0 \\ \pm i\rho(x) \otimes \sigma(y) & \text{if } x \in X\backslash X_0 \text{ and } y \in Y\backslash Y_0. \end{cases}$$

It is easy to check that $\tau_{\pm}$ is a homomorphism. The only case not straightforward is checking $\tau_{\pm}(xyx_1y_1) = \tau_{\pm}(xy)\,\tau_{\pm}(x_1y_1)$ when $x, x_1 \in X\backslash X_0$ and $y, y_1 \in Y\backslash Y_0$. Here

$$\begin{aligned} \tau_{\pm}(xyx_1y_1) = \tau_{\pm}(zxx_1yy_1) &= -\rho(xx_1) \otimes \sigma(yy_1) \\ &= -(\rho(x)\rho(x_1)) \otimes (\sigma(y)\sigma(y_1)) \\ &= -(\rho(x) \otimes \sigma(y))(\rho(x_1) \otimes \sigma(y_1)) \\ &= (\pm i\rho(x) \otimes \sigma(y))(\pm i\rho(x_1) \otimes \sigma(y_1)) \\ &= \tau_{\pm}(xy)\tau_{\pm}(x_1y_1). \end{aligned}$$

Evidently $\tau_{\pm}$ is irreducible as its restriction to $X_0 Y_0$ is irreducible. ∎

Remark. It follows from Proposition 8.3 that any irreducible character of XY reduces upon restriction to $X_0 Y_0$.

The results of Propositions 8.2 and 8.3 can be used to define inductively the characters $\phi_{\beta_1,\ldots,\beta_r}$ introduced in [22, (4.2) and (4.3)] and described in 2D. Indeed, if $\beta_1 \in \mathrm{DP}_a$ and $\beta_2 \in \mathrm{DP}_{n-a}$, then the character ϕ_{β_1,β_2} is $\phi_{\beta_1} * \phi_{\beta_2}$. Furthermore, $\phi_{\beta_1,\ldots,\beta_r} = \phi_{\beta_1,\ldots,\beta_{r-1}} * \phi_{\beta_r}$, and the characters can be obtained directly from Propositions 8.2 and 8.3. This provides an alternate method for treating the characters $\phi_{\beta_1,\ldots,\beta_r}$.

We now prove two lemmas which are used to analyse the case where $\phi_{\{n\}}$ remains irreducible when restricted to $\hat{S}_{n-a,a}$ and $\widehat{S_a \wr S_b}$ (with $n = ab$).

LEMMA 8.4. *Assume that $X \leqslant \hat{S}_n$ and that $\phi_\lambda | X$ is irreducible. Then $\phi_\lambda | X$ is not self-associated if and only if λ is odd and the image of X in S_n contains a λ-cycle.*

Proof. Assume that $\phi_\lambda | X$ is not self-associated. Then neither is ϕ_λ, and so λ is odd. Moreover ϕ_λ^+ and ϕ_λ^- differ only on λ-cycles, and so X must contain a λ-cycle (see 2A). The converse is also clear. ∎

LEMMA 8.5. *Assume hypotheses* (a)–(e) *above, so that $X \leqslant \hat{S}_a$ and $Y \leqslant \hat{S}_{n-a}$. Further assume that $\mu \in \mathrm{DP}_a$, $\nu \in \mathrm{DP}_{n-a}$, $\phi_\mu | X$ and $\phi_\nu | Y$ are both irreducible, X contains a μ-cycle, and Y contains a ν-cycle.*

(i) *If both $\phi_\mu | X$ and $\phi_\nu | Y$ are self-associated, or if neither is self-associated, then $\phi_{\mu,\nu} | XY = (\phi_\mu | X) * (\phi_\nu | Y)$, which is irreducible and self-associated.*

(ii) *If $\phi_\mu | X$ is self-associated while $\phi_\nu | Y$ is not self-associated, then $\phi_{\mu,\nu}^+ | XY$ and $\phi_{\mu,\nu}^- | XY$ are distinct associated irreducible characters of XY.*

Proof. Observe that the character values of $\phi_{\mu\nu} | XY$ given in [22, (4.2)] agree with those of $(\phi_\mu | X) * (\phi_\nu | Y)$ given in Proposition 8.2, and the result follows. ∎

We are now ready for the proof of Theorem 8.1.

Proof of Theorem 8.1. First assume that H is a maximal subgroup such that $\phi_{\{n\}} | \hat{H}$ is irreducible. If H is primitive, then (i) or (iv) holds by Theorem 6.3. If H is transitive yet imprimitive, then (iii) holds. Finally, assume that H is intransitive, so that $H = S_{n-a,a}$ for some $a < n/2$. Evidently $\phi_{\{n-a\},\{a\}}$ is a constituent of $\phi_{\{n\}} | \hat{H}$, and if n is odd, then both characters $\phi^{\pm}_{\{n-a\},\{a\}}$ are constituents. Therefore n must be even, and so (ii) holds.

We now prove the "if" part of the theorem. If H appears in (i) or (iv), then $\phi_{\{n\}} | \hat{H}$ is irreducible by 2B and Theorem 6.3. Next suppose that (ii) holds. It follows from 2D and 2E (second paragraph) that the only possible constituents of $\phi_\lambda | \hat{S}_{n-a,a}$ are of the form $\phi_{\{n-a\},\{a\}}$ (for if $\mu \in \mathrm{DP}_{n-a}$ and $\mu \leqslant \{n\}$, then $\mu = \{n-a\}$). As n is even, either both $\phi_{\{n-a\}}$ and $\phi_{\{a\}}$

are self-associated, or neither is self-associated, and so according to Lemma 8.5(i), $\phi_{\{n-a\},\{a\}} \mid \hat{S}_{n-a,a}$ is irreducible.

Suppose now that H is transitive yet imprimitive, so that $H = S_a \wr S_b$, with $n = ab$. We use induction on b to show that $\phi \mid \hat{H}$ is irreducible. There is nothing to prove when $b=1$, and when $b=2$, we may use the argument in the previous paragraph. So suppose that $b \geqslant 3$ and consider the subgroup $K = S_a \times (S_a \wr S_{b-1})$ of H and the subgroup $L = S_a \wr S_{b-1}$ of K. Clearly $K \leqslant S_a \times S_{a(b-1)}$, and we assume by induction that $\phi_{\{a(b-1)\}} \mid \hat{L}$ is irreducible.

Suppose first that a is even. Then by part (ii) of the theorem proved above, $\phi_{\{n\}} \mid \hat{S}_{a,a(b-1)} = \phi_{\{a\},\{a(b-1)\}}$ is irreducible. Moreover, $\phi_{\{a(b-1)\}} \mid \hat{L}$ is irreducible (by induction), and this character is not self-associated because $a(b-1)$ is even and L contains $a(b-1)$-cycles (see Lemma 8.4). In addition, $\phi_{\{a\}} \mid \hat{S}_a$ is not self-associated for the same reason. Therefore by Lemma 8.5(i), $\phi_{\{a\},\{a(b-1)\}} \mid \hat{K}$ is irreducible. Thus $\phi_{\{n\}} \mid \hat{K}$ is irreducible, and hence so is $\phi_{\{n\}} \mid \hat{H}$.

Suppose a is odd and b is even. As before, $\phi_{\{n\}} \mid \hat{S}_{a,a(b-1)} = \phi_{\{a\},\{a(b-1)\}}$ is irreducible. In this case, however, both $\phi_{\{a\}}$ and $\phi_{\{a(b-1)\}}$ are self-associated, and so we may reason as in the previous paragraph, applying Lemma 8.5(i) again.

To finish, suppose a is odd and b is odd. Here $\phi_{\{a(b-1)\}} \mid \hat{L}$ is irreducible and not self-associated, while $\phi_{\{a\}} \mid \hat{S}_a$ is self-associated. Thus by Lemma 8.5(ii), $\phi_{\{n\}} \mid \hat{K}$ is a sum of two distinct irreducibles, ψ_1, ψ_2, say. Thus the corresponding $\mathbf{C}\hat{K}$-module contains precisely two irreducible $\mathbf{C}\hat{K}$-submodules, V_1 and V_2, say. Now Lemma 8.5(ii) shows that $\phi_{\{n\}} \mid \hat{S}_{a,a(b-1)}$ is also a sum of two distinct irreducibles, whence V_1 and V_2 are also $\hat{S}_{a,a(b-1)}$-invariant. As $\hat{S}_{a,a(b-1)}$ is maximal in $\hat{S}_n$, and as V_1 and V_2 are not $\hat{S}_n$-invariant, it follows that an element in $h \in \hat{H} \setminus \hat{S}_{a,a(b-1)}$ does not fix V_1 and V_2. Consequently $\phi_{\{n\}} \mid \langle \hat{K}, h \rangle$ is irreducible, whence so is $\phi_{\{n\}} \mid \hat{H}$. The proof is complete. ∎

All parts of Theorem 1.1 have now been proven, using Theorem 5.1, Theorem 6.3, Theorem 7.1, and Theorem 8.1. (Note that the case from Theorem 5.1(iv) occurs in Theorem 1.1 only with $H = S_{n-1,1}$, for it is assumed in Theorem 1.1 that $\hat{H}$ is maximal in $\hat{S}_n$.)

9. The $\hat{A}_n$ Case

In this section we apply the results and arguments above to prove Theorem 1.2 for $\hat{A}_n$. Recall from 2B that the irreducible characters of $\hat{A}_n$ are $\eta_\lambda = \phi_\lambda \mid \hat{A}_n$ for λ odd and the two constituents η^i_λ $(i = 1, 2)$ of $\phi_\lambda \mid \hat{A}_n$ when λ is even. In particular, if λ is even, $\phi_\lambda \mid \hat{A}_n = \eta^1_\lambda + \eta^2_\lambda$, and as usual, η_λ

denotes either η_λ^i in this case. The following lemma is exploited several times in the proof of Theorem 1.2.

LEMMA 9.1. *Assume that* $K = S_{n-a,a}$ *or* $K = S_a \wr S_b$ *with* $n = ab$, *and that* $H = K \cap A_n$.

(i) *Assume that* $\phi_\lambda | \hat{K}$ *is irreducible. Then* $\eta_\lambda | \hat{H}$ *is irreducible if and only if either* λ *is even or* λ *is odd and* K *contains a* λ-*cycle.*

(ii) *If* $\phi_\lambda | \hat{K}$ *is irreducible and* $\eta_\lambda | \hat{H}$ *is irreducible, then* λ *is even,* $\phi_\lambda | \hat{K} = \chi_1 + \chi_2$, *with* χ_i *irreducible,* $\chi_1 \neq \chi_2$, χ_1 *and* χ_2 *are associates, and* $\chi_1 | \hat{H} = \chi_2 | \hat{H} = \eta_\lambda^1 | \hat{H} = \eta_\lambda^2 | \hat{H}$.

Proof. Part (i) is immediate from 2A, 2B, and Lemma 2.3. We now prove (ii). If λ is odd, then $\phi_\lambda | \hat{A}_n$ is irreducible, and hence $\phi_\lambda | \hat{H}$ is irreducible (since $\eta_\lambda = \phi_\lambda | \hat{A}_n$). But this forces $\phi_\lambda | \hat{K}$ to be irreducible, against our hypothesis. Thus λ is even. Now if $k \in \hat{K} \setminus \hat{H}$, then $(\eta_\lambda^1)^k = \eta_\lambda^2$ by Lemma 2.3. Consequently k interchanges $\eta_\lambda^1 | \hat{H}$ and $\eta_\lambda^2 | \hat{H}$ by conjugation, so they are both irreducible, and so $\phi_\lambda | \hat{H}$ is a sum of two irreducibles. Thus the same holds for $\phi_\lambda | \hat{K}$. Hence we may write $\phi_\lambda | \hat{K} = \chi_1 + \chi_2$, with χ_i an irreducible character of $\hat{K}$ and $\chi_i | \hat{H}$ an irreducible character of $\hat{H}$. By Lemma 2.3, $\chi_i(k') \neq 0$ for some $k' \in \hat{K} \setminus \hat{H}$. Since λ is even, $\phi_\lambda(k') = 0$. Therefore $\chi_1(k') = -\chi_2(k')$, which proves $\chi_1 \neq \chi_2$. Since $(\mathrm{sgn})\,\phi_\lambda = \phi_\lambda$, it follows that both χ_1 and $(\mathrm{sgn} | \hat{K})\,\chi_1$ are constituents of $\phi_\lambda | \hat{K}$, and hence $\chi_2 = (\mathrm{sgn} | \hat{K})\chi_1$. In other words, χ_1 and χ_2 are associates. In particular, $\chi_1 | \hat{H} = \chi_2 | \hat{H}$, and so the final equalities in the statement of (ii) are now clear. ∎

We begin the proof of Theorem 1.2 by proving the "only if" portion. So assume that H is a maximal subgroup of A_n and that $\eta_\lambda | \hat{H}$ is irreducible for some λ. If H is primitive, then we appeal to Theorem 6.4. This situation gives rise to cases (8)–(16) of Theorem 1.2. Note that $AGL_2(3) \cap A_9 \cong 3^2 : SL_2(3)$.

Suppose next that H is intransitive, so that $H = S_{n-a,a} \cap A_n$ with $a < n/2$. Let K be the subgroup $S_{n-a,a} \leqslant S_n$ containing H as a subgroup of index 2. If $\phi_\lambda | \hat{K}$ is irreducible, then K and λ appear in parts (2), (3), or (4) of Theorem 1.1. If K and λ occur in part (2) of Theorem 1.1 (Theorem 1.1.(2)), then by Lemma 9.1(i), it must be the case that $r = 0$, and this gives rise to Theorem 1.2(2). If K and λ occur in Theorem 1.1(3), then H and λ appear in Theorem 1.2(3). On the other hand, K and λ cannot occur in Theorem 1.1(4), for here $\lambda = \{n\}$ is odd and K does not contain an n-cycle (see Lemma 9.1(i)). We may assume therefore that $\phi_\lambda | \hat{K}$ is reducible, so that Lemma 9.1(ii) applies. In particular, λ is even. If λ has more than one corner, then arguments appearing in the proof of Theorem 5.1 using Theorem 4.3(ii) show that $\langle \psi_{n-2,2} | K, 1 \rangle_K = 0$, and hence [15] implies that $K=S_{n-1}$. But according to the branching rule (2.8), $\phi_\lambda \mid \hat{S}_{n-1}$ has at least three constituents when λ has at least two corners; this contradicts the fact that $\phi_\lambda \mid K$ has only two irreducible constituents, by Lemma 9.1(ii).

Thus λ has just one corner. If $\lambda=\{n\}$, then we must have n odd (since λ is even), and so part (4) of Theorem 2 occurs. We may assume therefore that $\lambda \neq \{n\}$, so that $l \geqslant 2$. If $a \geqslant 3$, then Corollary 2.2 implies that $\phi_\lambda \mid \hat{K}$ has at least two non-associated constituents $\phi_{\mu,\nu}$, $\phi_{\mu',\nu'}$. As $\phi_{\mu,\nu} \mid \hat{H} \neq \phi_{\mu',\nu'} \mid \hat{H}$, we reach the desired contradiction by Lemma 9.1(ii). Therefore $a \leqslant 2$. If $a=1$, then it must be the case that $\lambda_l \geqslant 2$, as we are assuming that $\phi_\lambda \mid \hat{K}$ is reducible (see Theorem 1.1(2)); thus H and λ appear in Theorem 1.2(1). Finally, we are left with the case in which $a=2$. Since $\phi_\lambda \mid \hat{K}$ is reducible, it follows from Theorem 1.1(3) that $\lambda_l \geqslant 2$. But then Lemma 4.2(i) implies that $\phi_\lambda \mid \hat{K}$ has norm at least 3, which violates Lemma 9.1(ii). This contradiction completes the analysis in the case where H is intransitive.

Now assume that H is transitive, so that $H=K \cap A_n$, where $K=S_a \wr S_b$ and where $n=ab$. First assume that $\phi_\lambda \mid \hat{K}$ is irreducible, so that K, λ appear in parts (5), (6), or (7) of Theorem 1.1. If K and λ appear in parts (5) or (6) of Theorem 1.1, then H and λ appear in parts (5) or (6) of Theorem 1.2. Consider now the case where K and λ appear in part (7) of Theorem 1.1. Here $\lambda=\{3,2,1\}$, which is odd, and so Lemma 9.1(i) implies that K contains a λ-cycle. However, $S_2 \wr S_3$ does not contain a λ-cycle. Therefore $K=S_3 \wr S_2$, and so H appears in part (7) of Theorem 1.2. Thus we can assume that $\phi_\lambda \mid \hat{K}$ is reducible. Therefore Lemma 9.1(ii) applies, and so λ is even. As in the intransitive case above, if λ has two or more corners, then the multiplicity of $\psi_{n-2,2}$ in $\phi_\lambda \bar{\phi}_\lambda$ is greater than 1, which forces K to be either 2-homogeneous or intransitive, a contradiction. We infer that λ has just one corner, and so $n=\frac{1}{2}l(l+1)+rl$, $\lambda=\{l+r, l-1+r, \ldots, 1+r\}$, and

$$\phi_\lambda(1)=\frac{2^{(n-l)/2}\left(\frac{1}{2}l(l+1)+rl\right)!}{\varepsilon_\lambda(l+r)!\,(l-1+r)!\cdots(1+r)!}\prod_{i<j}\frac{j-i}{2r+i+j}.$$

Since $\phi_\lambda \mid \hat{K}$ is reducible, $\lambda \neq \{n\}$ by Theorem 1.1(5), and hence $l \geqslant 2$. When $l=2$, we have $n=2r+3$ and $\phi_\lambda(1)$ is divisible by all primes p satisfying $r+3 \leqslant p \leqslant 2r+2$. Note that $r \geqslant 3$ since n is not prime, and so by Theorem 2.4, there is always such a prime p in this range (small values of r must be checked separately). But then $p \mid \eta_\lambda(1)$, yet p does not divide $|\hat{H}|$ as $p>n/2$, a contradiction. If $l \geqslant 3$, then $\eta_\lambda(1)$ is divisible by all primes between $2r+2l-1$ and n. Note that $2r+2l-1 \leqslant \frac{2}{3}n+2$, and so by Theorem 2.4, there is always such a prime in this range, provided $\frac{2}{3}n+2 \geqslant 29$, or $n \geqslant 41$. When $n \leqslant 40$, we check by hand that there is always such a prime, except when $n=6$, 9, or 10. If $n=10$, then $\lambda=\{4,3,2,1\}$, and due to our assumption that $\phi_\lambda \mid \hat{K}$ is reducible, $K=S_2 \wr S_5$. But then

$\eta_\lambda(1)^2 = 48^2 > \frac{1}{2}2^5 5! = \frac{1}{2}|\hat{H}|$, and hence $\eta_\lambda | \hat{H}$ is reducible (n.b., the sum of the squares of the faithful irreducible characters of $\hat{H}$ is bounded above by $\frac{1}{2}|\hat{H}|$). Similarly, if $n=9$, then $\lambda = \{4, 3, 2\}$, and $\eta_\lambda(1)^2 = 48^2 > 3^4 \cdot 2^3 = \frac{1}{2}|\hat{H}|$. This leaves the case $n = 6$, which is impossible in light of our assumption that $\phi_\lambda | \hat{K}$ is reducible.

To complete the proof of Theorem 1.2 we must show that $\lambda | \hat{H}$ is irreducible for all λ and H which occur in parts (1)–(16) of the statement. For the primitive groups in (8)–(16), we can quote Theorem 6.4. Moreover, since the imprimitive groups $S_a \wr S_b$ (with $n = ab$) contain an n-cycle, it follows from Theorem 1.1(5) and Lemma 9.1(i) that Theorem 1.2 holds in case (5). In the exact same way we see that Theorem 1.2 holds in cases (2), (3), (6), and (7). (Note that in case (6), $\{4, 3, 2, 1\}$ is even, and in case (7), $S_3 \wr S_2$ contains a $\{3, 2, 1\}$-cycle.) Now assume that H and λ are as described in part (1) of Theorem 1.2. Then $\phi_\lambda | \hat{S}_{n-1} = \phi^+_{\lambda/c} + \phi_{\lambda/c}$, where c is the bottom corner of λ. Evidently λ/c is odd as λ is even, and so $\phi^\pm_{\lambda/c} | \hat{A}_{n-1}$ is irreducible by 2B. However, $\hat{A}_{n-1}$ is precisely $\hat{H}$, whence $\phi_\lambda | \hat{H}$ is a sum of just two irreducibles. Consequently $\eta^i_\lambda | \hat{H}$ must be irreducible for $i = 1, 2$. Finally, take the case where H and λ are as in part (4) of Theorem 1.2. Here $\phi_{\{n\}} | \hat{S}_{n-a,a}$ is a sum of two associated characters $\phi^\pm_{\{n-a\},\{a\}}$, neither of which vanish on an element τ which is the product of an $(n-a)$-cycle in $\hat{S}_{n-a}$ with an a-cycle in $\hat{S}_a$ (see [22, (4.2)] or Proposition 8.2(iii)). Note that $\tau \in \hat{K} \backslash H$. Thus by Lemma 2.3, $\phi^\pm_{\{n-a\},\{a\}} | \hat{H}$ is irreducible. Consequently $\phi_\lambda | \hat{H}$ has just two irreducible constituents, both of which are $\eta^1_{\{n\}} | \hat{H} = \eta^2_{\{n\}} | \hat{H}$. This completes the proof of Theorem 1.2.

10. The Proof of Theorem 1.3

To prove Theorem 1.3, we first assume that $\hat{H}$ is a quasisimple proper subgroup of $\hat{A}_n$, that η_λ is a faithful irreducible character of $\hat{A}_n$, and that $\eta_\lambda | \hat{H}$ is irreducible. We see that H (the image of $\hat{H}$ in A_n) is simple, for $Z(\hat{H})$ acts as scalar matrices on the corresponding module, and so $Z(\hat{H}) \leqslant Z(\hat{A}_n)$. As in the proofs of Theorems 6.3 and 6.4, $|H| \geqslant 2^{n-4}$, and hence H and n appear in Proposition 6.1. If H acts transitively on $\{1, ..., n\}$, then it is easily checked that X is in fact primitive (this holds because the index of any non-maximal subgroup of H is always larger than the upper bound given on n). So, in this case, H is given in Theorem 6.4, and hence H appears in parts (4)–(8) in Theorem 1.3.

We can assume therefore that H is intransitive. Thus H is contained in a maximal subgroup K of A_n, and K is of the form $S_{n-a,a} \cap A_n$ or $S_{n/2} \wr S_2 \cap A_n$. In the first case, K and λ appear in parts (1)–(4) of Theorem 1.2. First consider (1), so that $\hat{K} = \hat{A}_{n-1}$. Then $\eta_\lambda | \hat{K} = \eta_{\lambda/c}$, where c is the bottom corner. However, $\lambda/c = \{l+r, l-1+r, ..., 1+r, r\}$, and hence $\eta_{\lambda/c}$ is reducible upon restriction to all proper subgroups of $\hat{A}_{n-1}$ by Theorem 1.2. Thus $H = K$, and so H appears in Theorem 1.3(1). Next assume that K and λ occur in part (2) of Theorem 1.2. Then $K = A_{n-1}$ and $\eta_\lambda \mid \hat{K} = \eta_{\lambda/c}$, where again c is the bottom corner and $\lambda/c = \{l, l-1, ..., 3, 2\}$. If $H = K$, then H appears in Theorem 1.3(1). If $H \neq K$, then it follows from the previous argument that $A_{n-2} = H = S_{n-2,1} \cap K$, and it must be the case that λ/c is even, as stipulated in Theorem 1.2(1). Consequently λ is even, and so H appears in Theorem 1.3(2). Next assume that K and λ occur in part (3) of Theorem 1.2. Here $K \cong S_{n-2}$, and $\tilde{K} \cong \hat{S}_{n-2}$. Moreover, Lemma 4.2 and Proposition 8.3 show that $\eta_\lambda \mid \hat{K} = \phi_\mu$, where $\mu = \{l, l-1, ..., 3, 1\}$. According to Theorem 1.1, ϕ_μ remains irreducible upon restriction to a proper subgroup only if μ is odd, and hence λ is even. Moreover, the only maximal subgroup of $\hat{K}$ to which ϕ_μ restricts irreducibly is $\hat{A}_{n-2}$, and hence $H \leqslant A_{n-2}$. However, according to Theorem 1.2, the character η_μ of $\hat{A}_{n-2}$ never restricts irreducibly to a proper subgroup of $\hat{A}_{n-2}$, and hence it must be the case that $H = A_{n-2}$. Once again, H appears in part (2) of Theorem 1.3. Now suppose that K and λ occur in Theorem 1.2(4), so that n is odd. Consider first the case in which $a = 1$, so that $H \leqslant A_{n-1}$. Here $\eta_{\{n\}} \mid \hat{A}_{n-1} = \eta_{\{n-1\}}$, and since $n-1$ is even, it follows from Theorem 1.2 that H must be transitive on $n-1$ points. The case in which $H = A_{n-1}$ appears in Theorem 1.3(3). So we can assume that $H < A_{n-1}$. Since H is transitive in A_{n-1}, we deduce (as we did in the first paragraph of this section) that H is in fact primitive. Therefore H appears in Table II. Since H is simple and $n-1$ is even, the only possibilities are $H = A_5$, $n-1 = 6$, and $H = M_{12}$, $n-1 = 12$. Consequently H and λ appear in parts (9) or (10) of Theorem 1.3. We can assume now that $a \geqslant 2$. According to the proof of Theorem 8.1 in Section 8, $\phi_{\{n\}} \mid \hat{S}_{n-a,a} = \phi^+_{\{n-a\},\{a\}} + \phi_{\{n-a\},\{a\}} = (\phi_{\{n-a\}} * \phi_{\{a\}})^+ (\phi_{\{n-a\}} * \phi_{\{a\}})^-$. Moreover, according to the remark after the proof of Proposition 8.3, each irreducible character of $\hat{S}_{n-a,a}$ reduces when restricted to $\hat{A}_{n-a}\hat{A}_a$. It now follows that the restriction of $\phi_{\{n\}}$ to $\hat{A}_{n-a}\hat{A}_a$ contains at least four irreducible constituents. Consequently, for $i = 1, 2$, the restriction of $\eta^i_{\{n\}}$ to $\hat{A}_{n-a}\hat{A}_a$ has two irreducible constituents. In particular, $\eta^i_{\{n\}} \mid \hat{H}$ is reducible, and so no example arises here.

Finally, suppose that $K = S_{n/2} \wr S_2 \cap A_n$. Here K appears in (5), (6), or (7) in Theorem 1.2. As $S_3 \wr S_2$ is solvable, we can eliminate part (7). If K occurs in (6), then it must be the case that $H \cong A_5$. But here $\lambda = \{4, 3, 2, 1\}$, whence $\eta_\lambda(1) = 48$, which is not the degree of an irreducible character of a covering group of A_5. This leaves the case where K occurs in (5), and here $\lambda = \{n\}$. However, by using the remark after the proof of Proposition 8.3, we see that $\phi_{\{n\}} \mid \hat{A}_{n/2}\hat{A}_{n/2}$ is reducible. Since n is even, $\phi_{\{n\}} \mid \hat{A}_n = \eta_{\{n\}}$, and hence $\eta_{\{n\}} \mid \hat{A}_{n/2}\hat{A}_{n/2}$ is also reducible. Therefore $\eta_{\{n\}} \mid \hat{H}$ is reducible. We have therefore proved the only if portion of Theorem 3.1.

To prove the if portion, note that all entries in Theorem 1.3 have appeared already in Theorem 1.2, except for parts (2), (9), and (10). To handle part (2), we observe that $\eta_{\{l, l-1, ..., 1\}}$ restricts irreducibly to $\hat{S}_{n-2,2} \cap \hat{A}_n$ as $\phi_{\{l, l-1, ..., 3, 1\}}$ (see part (3) of Theorem 1.2), which in turn restricts irreducibly to $\hat{A}_{n-2}$ as $\eta_{\{l, l-1, ..., 3, 1\}}$ (note that $\{l, l-1, ..., 3, 1\}$ is odd as $\{l, l-1, ..., 3, 2, 1\}$ is even). To handle parts (9) and (10), use the branching rule and parts (9) and (16) of Theorem 1.2.

This completes the proof of Theorem 1.3.

References

1. M. Aschacher, Subgroup structure of finite groups, *in* "Proceedings, Rutgers Group Theory Year, 1983–1984" (M. Aschbacher *et al.*, Eds.) pp. 35–44, Cambridge Univ. Press, London/New York, 1984.
2. M. Aschbacher and L. L. Scott, Maximal subgroups of finite groups, *J. Algebra* **92** (1985), 44–80.
3. A. Bochert, Über die Zahl der vershiedenen Werthe, die eine Function gegebener Buchstaben durch Verstauschung derselben erlangen kann, *Math. Ann.* **33** (1889), 584–590.
4. P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.
5. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Oxford Univ. Press, London/New York, 1985.
6. B. N. Cooperstein, Minimal degree for a permutation representation of a classical group, *Israel J. Math.* **30** (1978), 213–235.
7. L. Duhuai and B. G. Wybourne, The symmetric group: Branching rules, products of plethysms for spin representations, *J. Phys. A* **14** (1981), 327–348.
8. L. Gatteschi, Un perfezionamento di un teorema di I. Schur sulla frequenza dei numeri primi, *Boll. Un. Mat. Ital.* (3) **2** (1947), 123–125.
9. G. D. James, "The Representation Theory of the Symmetric Group," Springer-Verlag, New York/Berlin, 1978.
10. W. Leveque (Ed.), "Reviews in Number Theory," Vol. 1, No. A40-7, p. 134, Amer. Math. Soc., Washington, DC, 1974.
11. M. W. Liebeck and J. Saxl, On the orders of maximal subgroups of the finite exceptional groups of Lie type, *Proc. London Math. Soc.* (3) **55** (1987), 299–330.
12. M. W. Liebeck, C. E. Praeger, and J. Saxl, On the O'Nan–Scott reduction theorem for finite primitive permutation group, *J. Austral. Math. Soc.* **44** (1988), 389–396.
13. A. O. Morris, The spin representation of the symmetric group, *Proc. London Math. Soc.* (3) **12** (1962), 55–76.
14. A. O. Morris and A. K. Yaseen, Decomposition matrices for spin characters of symmetric groups, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 145–164.
15. P. M. Neumann, Generosity and characters of multiply transitive permutation groups, *Proc. London Math. Soc.* **31** (1975), 457–481.
16. B. E. Sagan, Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley, *J. Combin. Theory Ser. A* **45** (1987), 62–103.
17. J. Saxl, The complex characters of the symmetric groups that remain irreducible in subgroups, *J. Algebra* **111** (1987), 210–219.
18. I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.

19. I. SCHUR, Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen. I., *Sitzungsber. Preus. Akad. Wissen.* (1929), 125–136.
20. L. L. SCOTT, Representations in characteristic p, *in* "The Santa Cruz Conference on Finite Groups" (B. N. Cooperstein *et al.*, Eds.), pp. 319–331, American Mathematical Society, Providence, RI, 1980.
21. G. M. SEITZ, Overgroups of irreducible linear groups, *in* "Proceedings, Rutgers Group Theory Year, 1938–1984" (M. Aschbacher *et al.*, Eds.), pp. 95–106, Cambridge Univ. Press, London/New York, 1984.
22. J. R. STEMBRIDGE, Shifted tableaux and the projective representations of the symmetric group, *Adv. Math.* **74** (1989), 87–134.
23. A. WAGNER, An observation on the degrees of projective representations of the symmetric and alternating groups over an arbitrary field, *Arch. Math.* **29** (1977), 583–589.
24. D. R. WORELY, "A Theory of Shifted Young Tableaux," Ph.D. thesis, MIT, 1984.